Persistent instability in polarized opinion formation and collective decision-making

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Abstract

In this article we present a simple model of opinion formation and collective decision making, which identifies the conditions for unstable, non-linear outcomes of these processes. We propose a simple modification of the commonly used model of French from 1957: social actors change their influence in order to avoid a future outcome they do not like. The analysis of a simple, *3*-member group shows that the social structure of the influence process has an important impact on the likelihood of persistent instability: one condition is polarization, a second is the presence of a bouncing subgroup or actor. The third condition is the intolerance for a discrepancy between the future outcome and the current opinion. In a complex 6-actor group computer simulations reveal very complex oscillations for intermediate levels of intolerance. The likelihood of these oscillations is discussed, as well as its consequences for the predictability of opinion formation and collective decision-making conditions.

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1. Introduction

“We live in a highly non-linear world,” Jay Forrester (1987) puts forward in an early paper on non-linearity in social systems. There is a vast collection of theoretical and empirical research on instability, non-linearity and chaotic behavior in social systems. The claims are simple and straightforward: very simple feedback mechanisms induce unpredictable behavioral outcomes. In economics, studies have been performed on business cycles containing multiple modes of behavior, which even exhibit deterministic chaos. The amplitudes of business cycles change over time and have a periodicity which is not constant. These instabilities have obvious and important consequences for macro-economic policy and economic forecasting.

But also on a smaller scale, non-linearity can be present in social and economic behavior. Economists Baumol and Behabib (1989) provide an example of two negotiators who do not know their reaction function on proposals. If they react on the outcomes reached in subsequent negotiation rounds, there is a theoretical possibility that the negotiations do not follow a clear path, but that the trajectory is better described as a complex oscillation of outcomes. Both parties will break down the negotiations, suspecting each other of sabotage, and an agreement will never be reached. It is a hypothetical example of a collective decision-making trap, not induced by the unwillingness of actors to reach an agreement, but by their inability to understand the structure of the social situation.

Where Baumol and Benhabib aim at a refutation of common predictions of economist models, Forrester’s goal is to construct more realistic models of social phenomena. According to Forrester, economic research should inform us about the ‘boundary conditions’ of non-linear behavior in social systems: under what circumstances is it likely that unpredictable, non-linear patterns occur? The empirical question whether these patterns occur in economic data seems to be resolved. Economists have found ample evidence for complex non-linear patterns, but much less evidence for deterministic chaos in empirical time series (Day 1992).

In the fields of political science and public administration chaos-theory and non-linear dynamics have become a major topic. Scholars suggest that policy making and collective decision-making sometimes produce unstable and unpredictable outcomes. Evidence, however, is much less hard than in economics. Three different classes of evidence for non-linear behavior exist. In the first place, anecdotes are described, suggesting complex and cyclic outcomes of highly complex policy processes. The lack of understanding of exact mechanisms in these cases is transformed to statements about alleged chaotic behavior.

In the second place, textbook systems of equations known to exhibit chaotical behavior are translated and interpreted into politics and policy making. Most models that show instability, non-linearity or even chaos are specifically designed to produce such outcomes. They are carbon-copied models, mostly originally designed for applications in natural sciences. These models are translated into the world of political phenomena by assuming that chaos and non-linearity is a basic feature of political life. Consequently, the discussion turns into a controversy: either political behavior is unstable and reflects non-linear processes, or political behavior is stable.

In the third place, large empirical time-series of political outcomes are analyzed applying statistical tools for detecting deterministic chaos. The latter type of evidence is scarce, but more compelling. Research shows that outcomes of collective decision making exhibit characteristics of non-linear dynamics. Tests for non-linearity or even chaotic behavior are developed and applied to time series data on political variables, especially on voting behavior. Richardson (1992) applies a spatial correlation test on two data sets. She shows that U.S. presidential popularity from 1953 though 1990 displays mainly random variation. However, a data set on nation state naval power over the past 500 years displays a chaotic process. Coleman (1993) applies an entropy measure and shows that voting cycles in the presidential elections in Alabama from 1916 through 1988 suggest characteristics of chaotic change. Only exceptional studies combine theoretical modeling with empirical analysis. Brown (1993) shows that electoral choices in the 1964 Johnson—Goldwater election are driven by non-linear processes. He describes the processes with the help of a formal model on partisan competition.

Our problem with most approaches is that they are ad-hoc and not well-founded in commonly used models of opinion formation and collective decision making. We would rather arrive at the theoretically more compelling deduction of (boundary) conditions under which we expect unstable behavior in opinion formation and collective decision-making. Such an analysis requires the use of a simple, commonly used model. The method of decreasing abstraction is the best way to approach the problem: we examine the effect of adding a simple, plausible assumption to a commonly used standard model of opinion formation. Subsequently, we explore the set of possible outcomes. The simple model enables us to relate the outcomes to the relevant boundary conditions.

We depart our analysis from a social influence model, applied on opinion formation and collective decision-making. Social actors adapt their opinions, or policy positions, on substantive issues over time. They do so subject to the influence of other actors. This model was originally developed in the late 1950s for the study of power, social influence and opinion formation in *N* person groups. The model served the interests of French (1956) and Harary (1959) in explaining the emergence of stable group standards and group norms. For Abelson (1964: 153) this result was rather troublesome because he tried to explain the existence of polarized communities with distinct cleavages. Abelson modeled the influence process as a *continuous change* in opinions and used differential equations, where actors in French's model adapt their opinion in different time-steps. Central to the model is the weight matrix, or contact matrix. This matrix reflects the institutional structure of influence in the social system and can be conceived of as a network of influence relations.

Later, in the 1990s, French's model was applied to collective decision-making in order to explain the outcomes of collective decision-making in national policy domains (Stokman & Van den Bos 1992). The conceptual difference between opinion formation and collective decision-making is small. Models are even functionally equivalent. This is not a coincidence, because the concept of 'power' forms the basis for social influence. French arrived at a power index and called his model a "theory of social power". His index was based on the combined relations and individual power base of group members. Abelson developed his simulation model to study community referendum controversies, applied to the fluoridation issue in local American communities. His aim was "representing the total political process in a community during a campaign on a local issue."

Surprisingly, the dynamic standard models, all of which are based on the same assumptions regarding social influence, fail to produce persistent instability on the individual as well as the collective level. We propose an extremely simple explanation of persistent instability. Our explanation adds one plausible assumption to the social influence models. This is the assumption that social actors change their influence in order to avoid a future outcome they do not like. We assume that social influence depends upon the discrepancy between expected outcome and the current opinion or policy position. The absence of this feedback effect is a special case of our generalized model: under specific theoretical conditions no feedback will occur and stable outcomes are guaranteed.

In section 2 we shortly describe the standard model of French from 1957 and give two examples of the inherent stability of this model. We pay specific attention to the social structures, which gives rise to persistent instability when the standard model is modified. In section 3 we introduce our modification of the standard model. Subsequently, we employ a combination of deductive analysis and computer simulations to study the patterns of behavior that may result from the feedback effect. We explore the boundary conditions under which our model produces different unstable patterns in the outcomes of collective decision making. In section 4 we replicate our analysis for a social system that is more complex. The paper closes with a discussion and conclusion in section 5.

2. The standard model

In 1954 French developed a simple, formal model of opinion formation. His model describes opinion formation as a process of the gradual emergence of compromise. More precisely, actors' opinions change over time driven both by the prior distribution of opinions and by an influence matrix that formalizes the degree to which actors can impose their opinions on each other. Technically, the initial opinion of a group member *i* at time *t* is expressed as a *real* number $x_i(t)$ that varies on a one-dimensional scale. For convenience, we assume in our implementation of French's model that opinions are bounded in the unity interval, i.e. $0 \le x_i(t) \le 1$. A group member adapts his opinion in every round *t* under the influence of the opinions of other group members *j*. A weight matrix **Y** defines the strength of the influence imposed by every group member ego on every other member alter. In the following, we use French's standard model to identify conditions for unanimity on basis of the distribution of initial opinions and the influence weight matrix.[1]

[1] If unanimity will be reached, the model predicts (a) the exact outcome of the unanimity opinion; and (b) the number of influence rounds required for unanimity. Of course, these predictions depend upon the specific scaling assumptions made.

2.1 The dynamics of opinion formation

French's standard model formalizes the new opinion of actor *i* as a weighted average of the opinions of those members by whom the focal actor is influenced, including his own opinion.[2] Formally,

[1] $$x_i(t+1) = \sum_{j \in N} y_{ij} x_j(t).$$

In equation [1], the influence weights y_{ij} scale the strength of the influence that *j* imposes on i. To scale relative influences between actors, the model normalizes the rows of the weight matrix. Accordingly, for member *i*, the total amount of incoming influence equals unity, i.e. $\sum_{j \in N} y_{ij} = 1$, where *N* is the set of group members. For simplicity, we use the symbol *N* also to denote group size. It is important to note that in this baseline model, influence weights y_{ij} are static, i.e. they do not change over time.

French's model interprets opinion formation in a group in terms of the operation of a "force field" between members. In this, the model implicitly assumes that all group members have full information on the opinions and of the relative influences in the group. The force field induces changes in the opinions of group members during a sequence of time steps. Opinions are assumed to be in a state of equilibrium. If group members *i* and *j* disagree, the strength of the force between them is proportional to two variables. Firstly, the strength is proportional to the level of their disagreement at the outset of the influence round *t*, e.g. $|x_i(t) - x_j(t)|$. Secondly, the force is proportional to their relative influences on each other, y_{ij} and y_{ji}. Moreover, the force is directed towards the more influential one of the two group members. The standard model assumes that group members adapt their position in such a way that the *net* force, simultaneously exercised by all other group members, is equal to zero. Thus, for each group member, the force field results in a new equilibrium opinion for each influence round.

The weight matrix affects the behavior of group members in a straightforward way. The most simple case arises when all group members exert equal influence on all other group members, where $y_{ij} = 1/N$ for all pairs *i* and *j*. In that situation, the group will converge to unanimity in one influence round, because the new opinion $x(t+1)$ is equal for actors, where the unanimity opinion of the group is the average opinion of all group members, i.e. $x(t+1) = \frac{1}{N} \sum_{j \in N} x_j$.[3] In this simple case, the position on the opinion scale where the

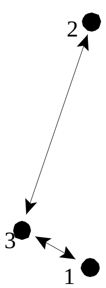
[2] We model the influence process as a time-discrete process. We assume that self-weights of actors are non-zero, i.e. $0 < y_{ii} \leq 1$. In Abelson's model, in which self-weights are zero, the influence process nevertheless converges because he models it as a continuous process, using differential equations.

[3] If the opinions in the standard model are interpreted as policy positions, the model would predict an outcome of group decision making which is the average position of all group members – of course under the

value of the unanimous group opinion is located depends solely upon the initial distribution of opinions. In the more general case, with unequal weights y_{ij}, the value of the unanimous group opinion depends additionally upon the structure of the weight matrix **Y**. In any case, the group opinion will be between the values of the opinions of the two most extreme group members.

Changes in opinions are not necessarily more complex if we assume some heterogeneity in the weight matrix **Y**. Figure 1 illustrates this assertion. The arrows in the figure represent the structure of the weight matrix and point out that not all three members of the group are able to directly reach all other group members. Members 1 and 3 are disconnected, but yet able to reach each other *indirectly*. The figure displays the results of a simulation of changes in opinion within the group. We assume initial opinions of $x_1(t_1) = 0$, $x_2(t_1) = 0.1$ and $x_3(t_1) = 0.9$. Figure 1 charts opinions on the vertical axis, while the horizontal axis represents influence rounds.

Figure 1 shows how the opinion formation process converges within a few iterations. The thick line represents the average position of the three group members in each influence round. In every influence round, the positions of the members move closer to each other and to average group opinion, until virtual unanimity is reached in iteration 5. This example demonstrates that stability is an important feature of the standard model of opinion formation. Stability arises at two different levels of analysis: (a) at the group-level, where monotonic convergence occurs towards one unanimous group opinion; (b) at the individual level, where all group members quickly adopt a stable opinion.



$$Y = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

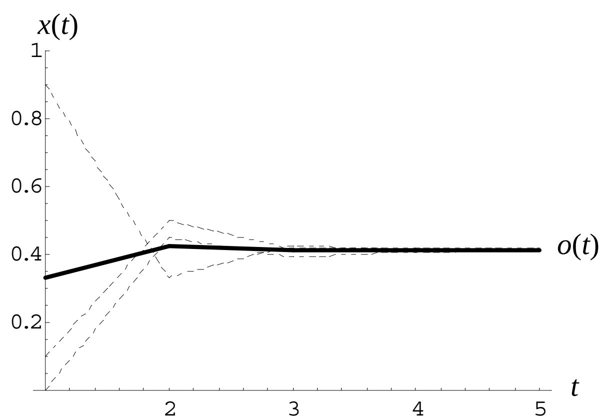


Figure 1. Unanimity in a *3*-person group

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conditions that the influence weights y_{ij} all are equal to one. In fact, the average policy position is used in models of collective decision making as a variant to Duncan Black's median voter theorem. For example, the two-stage model of collective decision-making, developed by Stokman and Van den Bos (1991) is functionally equivalent to French's standard model of opinion formation.

2.2 Conditions for group unanimity

French (1956), followed by Harary (1959) and Abelson (1964) were especially interested in the group level. They analyzed the model in order to find the necessary and sufficient conditions for a monotonic convergence towards unanimity in *N*-person groups with divergent initial opinions. In this literature, unanimity is defined as a situation where all opinions converge to one stable opinion *o*. Formally, the process can be characterized by $\lim_{t \to \infty} x_i(t) = o$ for all members *i* of the group. Eventually, there is no variation in the distribution of opinions under unanimity.

French showed that the form of the weight matrix **Y** is crucial for unanimity in opinion formation. He found that group cohesiveness, i.e. the abundance of influence relations, fosters convergence to unanimity. The more the group members are connected in the weight matrix, the faster the convergence. French coined the resulting force towards unanimity the "funneling effect" of increasing group cohesion. Moreover, for this model a necessary and sufficient condition for convergence on unanimity can be identified. Unanimity occurs if and only if at least one group member is directly or indirectly connected to all others. In other words, at least one group member can influence all others. Abelson (1964: 145) also detected the funneling effect in his continuous reformulation of the standard model. He related the effect to the concept of 'matrix compactness.' A contact matrix (the weight matrix) is compact if at least one group member is able to reach all others. When the weight matrix is compact, a universal equilibrium is the unique and stable equilibrium outcome of opinion formation on the group level.

2.3 Conditions for group polarization

Stability does not require unanimity. Stability may also occur in form of persistent polarization of stable opinions. Under persistent polarization, the process of opinion formation results in a stable situation, in which ultimately at least two different opinions emerge. Clearly, in this case there is no group opinion *o* to which every member adheres eventually. To define what the group opinion is under persistent disagreement, we follow models of political decision making and assume that there is in every influence round a final "voting phase", in which the individual opinions result in a group opinion: a norm or collective outcome (Stokman and van den Bos 1992). More technically, we assume that the group opinion is a weighted average of the opinions of all group members, where the weights indicate the constant relative "voting power" v_i of an actor *i*. The sum of these weights is constant at unity. More precisely, we assume that at the end of every influence round *t*, the group outcome $o(t)$ is computed according to equation [2].

[2] $$o(t) = \sum_{i \in N} v_i \; x_i(t)$$

Equation [2] implies that under persistent polarization, there is still convergence towards one stable group opinion *o*, while the standard deviation of individual opinions is larger than zero.

Persistent polarization will obtain if and only if "there is no individual who can eventually reach everybody" (Abelson 1964, 145), i.e. if the matrix is diffuse (Abelson uses the term non-compact). There are at least two very specific situations in which this condition is satisfied.[4] The first situation is the presence of *at least one isolated group member*. An *N*-person group will get polarized if *one* or more group members are isolated from the network.. An actor is isolated, if he can neither influence any other in the network, nor be influenced by any other group member. The reason for polarization is then simply that an isolated group member *i* will stick to his individual opinion regardless of the opinions of all group members *j* (since $y_{ij} = 0$ for all *j*). The other group members are not affected by the opinion of the isolated group member. Obviously, this first condition is trivial.

The second situation in which an *N*-person group gets polarized is more informative. This condition is the *presence of at least two stubborn actors*. Any other member of the group cannot influence a stubborn actor, but the stubborn actor imposes influence on at least one other group member. In other words, the stubborn actor is weakly connected to at least one other group member. Group members *i* and *j* are weakly connected if group member *i* is able to influence *j*, while *j* is unable to yield influence on member *i*. The presence of two stubborn actors yields a sufficient condition for polarization, because at least the positions of these two actors will never shift away from their initial opinions[5].

$$x(t=1) = (0.9,0,0.1) \qquad Y = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}$$

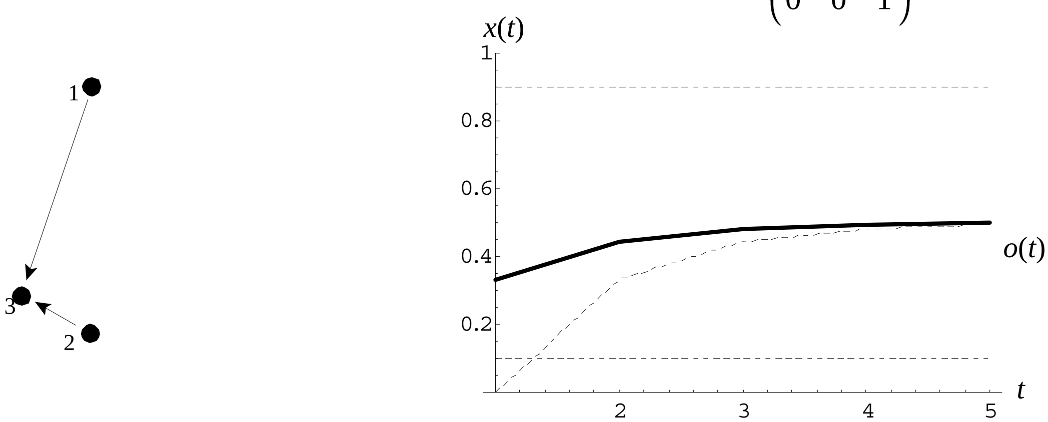

Figure 2. Polarization in a *3*-person group

[4] The conditions refer to the characteristics of individual group members but can be easily extended to the characteristics of subgroups within the network.
[5] In the presence of one stubborn actor a group will attain unanimity in a peculiar way: in the end, the process of opinion formation will always converge towards the opinion of the stubborn actor.

Figure 2 illustrates the effect of the presence of two stubborn actors in a three-person group. For simplicity, we assume equal voting power v=1/3 for all actors. The same group members are involved as in figure 1 with the same opinions. However, in contrast to figure 1, members 1 and 3 are ‘stubborn,’ i.e. they cannot be reached by any other group member. The influence weights for access of actor 2 to actors 1 and 3 are set to zero in the weight matrix. All group members still are weakly connected, but now the condition for polarization is present. Figure 2 shows how opinions do not converge in this situation. From the first influence round on, only actor 2 gradually adjusts his opinion towards the average of the opinions of actors 1 and 3. His opinion shifts from 0 to 0.50 over time, until virtual stability is reached in iteration 5. As a result, the group opinion o shifts from the initial group average (0.33) towards the average of the opinions of the stubborn actors 1 and 3 (0.50).

2.4 Persistent stability at the individual level

Group polarization does not preclude stability of individual opinions. Figure 2 shows that in our example group members 1 and 3 remain at their original opinions, while actor 2 rapidly converges to a stable opinion of 0.5.

Harary (1959) showed that the stability of individual opinions is a general property of the underlying model. Using the theory of Markov chains, Harary proved that “[i]n every group, regardless of its power structure or its initial opinion distribution, each member reaches a stable final opinion” (Harary, 1959: 180). More formally, for every group member j it holds that $\lim_{t\to\infty} x_j(t) = c_j$, where $0 \le c_j \le 1$ is a constant. Individual stability in opinions implies that the group opinion $o(t)$ will ultimately converge to a stable value o. Accordingly, monotonic convergence of the group opinion occurs both under unanimity and under polarized opinion formation.[6]

To conclude, our analysis of the standard model shows that the linearity of the influence mechanism in this model rules out the existence of unstable, cyclical or even chaotic patterns of opinion formation or collective decision-making. Persistent polarization is the most extreme pattern of opinion formation. We consider this property of the standard model unsatisfactory. The reason is not that we deny that convergence or polarization may often occur in empirical cases of opinion formation or collective decision-making. Yet, empirical studies indicate that there is nothing to guarantee stability in these processes. Accordingly, we should obtain the theoretical tools necessary to allow us to identify *conditions* and *mechanisms* under which instability may occur. The standard model fails to serve this purpose with its inherent property of convergence on the individual level. Accordingly, in the following section we present a simple extension of the standard model that allows us to address the conditions for and forms of instability in group opinion formation and decision-making.

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[6] In the case of polarization, the spread of the distribution σ_x (t) approaches a constant value larger than one. In the case of unanimity, the standard deviation of opinions σ_x (t) approaches zero as time increases.

3. A non-linear model of opinion formation

Stability of the opinion formation process is inherent to the standard French / Harary / Abelson model we discussed in the previous section. This may be interpreted as a consequence of the assumption of full information underlying the model. The model assumes that throughout the opinion formation process, actors operate on basis of a constant, commonly known weight matrix **Y**. With this assumption, force fields among the group members remain constant from round to round, and subsequent rounds of opinion dynamics constitute nothing more than gradual convergence to the unique equilibrium distribution imposed by the weight matrix and the initial opinions.

We feel that it is an implausible property of the standard model that it combines the full information assumption with the notion of gradual convergence to an equilibrium outcome. If group members are fully informed about others' opinions and influence weights from the outset on, they do not need an adaptive, time-discrete process in order to reach unanimity in opinion, or to get polarized. In the first round, they could all agree on the final outcome. However, we believe that the model of repeated mutual influencing is a plausible description of opinion formation when it is combined with the notion that actors' knowledge of the influence structure in the group may be incomplete or only preliminary. In that case, every influence round may be seen as boundedly rational "myopic optimization" on basis of the present situation. Actors are aware that after the influence round they receive new information on the basis of which they again might need to adapt their influence strategies.[7]
Clearly, this view introduces the possibility of more complex dynamics than exhibited by the standard model. To analyze these dynamics, we introduce in the following a simple model of myopic adjustment of influence strategies.

3.1 Expected frustration and dynamic adjustments of the weight matrix

At the outset of a decision round, actors may myopically assume that the outcome of the next round will be identical to the final outcome of the voting phase of the preceding round. Accordingly, they expect that they will experience at least some frustration after the round, because their current opinion diverges to some degree from the prior and expected decision outcome. In other words, actors make a myopic prediction of the utility loss they will suffer. In a process of opinion formation, this loss arises because an actor may have a deviant position in the group. In a process of collective decision-making, utility loss arises because the expected outcome differs from the most preferred outcome. Formally, the expected frustration in round $t+1$ of group member j is the absolute

[7] This notion of myopic optimization is increasingly used in game theoretical modeling as a 'low rationality' alternative to the standard assumption of perfect rationality. Models of so-called 'fictitious play' or 'belief learning' describe myopic optimization as the strategy to optimally respond to expected outcomes solely based on past experience. Unlike fully rational actors, myopic optimizers are naive in the sense that they do not try to anticipate future changes in others' behavior (Fudenberg and Levine 1998). While our model is not strictly a model of fictitious play, it clearly draws on the notion of myopic optimization used in this literature.

difference $|x_j(t) - o(t)|$ between the group outcome expected at the outset of t+1, $o(t)$, and the final opinion of *j* of the previous round *t*, $x_j(t)$.

3.2 Non-linear dynamics of opinion formation

It is reasonable to suppose that the expectation to get frustrated by the group outcome will drive actors to invest some effort in changing their relative weights in the influence process. More precisely, we assume that group members change their weights in the following influence round proportional to their expected level of frustration. One might interpret this change in relative weights as a change in issue salience, a variable that plays a central role in many models of collective decision-making. [8] If a group member expects a satisfactory outcome, he will put no additional effort in achieving this outcome. At the same time, if the expected outcome deviates from his opinion or preference, the group member will raise his level of salience for the issue at stake.

Our assumption of changing influence weights introduces a feedback effect into the standard model. We assume that the effort group members invest to influence others increases with the difference between the decision outcome and the present individual opinion. This effort in turn changes the opinions, which affect future investments in effort, etc. Clearly, the resulting feedback effect might induce endogenous instability. The assumption implies that the weight matrix **Y** in French's model changes over time, dependent on previous weights and expected frustration.

To model the dynamics of influence weights, we assume that the weights partially depend on expected frustration, and partially on a latent static influence structure y_{ij}. Equation [3] formalizes how dynamics influence weights for the new influence round *t*+1, y^*_{ij} (t+1), are computed.

[3]
$$y^*_{ij}(t+1) = \frac{y_{ij}\left|x_j(t) - o(t)\right|^{\tau}}{\sum_{k \in N} y_{ik}\left|x_k(t) - o(t)\right|^{\tau}}$$

The numerator in equation [3] expresses our central substantive assumption. The larger the difference between expected outcome and the present opinion of actor *j*, the larger the influence weight that actor *j* imposes on actor *i*. This effect is scaled with an *intolerance parameter* τ. Broadly, the larger τ, the more difference it makes for the group member to experience a discrepancy between the expected group opinion and his own opinion. The resulting influence weight is subsequently modified by the static component of *j*'s power

[8] In fact, Abelson incorporates into the standard model the concepts of: (a) actor preferences and (b) their interests in substantive policy issues. Abelson acknowledges the personal advice of James Coleman who, at that moment, was working on models of political exchange.

over i, y_{ij} . Finally, the denominator in Equation [3] guarantees that the row-sum of all influence weights is kept constant at one.

We combine our model of dynamic influence with the standard model as follows. At the outset of every decision round actors use equation [2] to calculate the expected group outcome of round *t* on basis of the final opinions of the preceding round *t*. Subsequently, actors adapt influence weights according to equation [3] in order to minimize expected frustration in round *t*. This results in a new weight matrix $\mathbf{Y}^*\mathbf{(t)}$ that drives the influence dynamics in round *t*+1. The influence dynamics as such proceed according to the standard model, as expressed by equation [4] below.

$$x_i(t+1) = \sum_{j \in N} y^*_{ij}(t+1)\, x_j(t). \qquad [4]$$

Clearly, the plausible assumption that expected frustration may change influence weights, may generate qualitatively different dynamics in the process of opinion formation. In the next sections we search for the simplest possible social system that could exhibit unstable behavior on basis of the simple feedback that we introduced into the standard model.

3.3 Unstable behavior in a three-person group

We define *unstable behavior* as non-monotonic change in opinion over time. We define *persistent instability* as persistent non-monotonic change in opinion ad infinitum. In our model, a group member adjusts his influence weights as soon as he discovers that the expected outcome differs from his present opinion. If the discrepancy is large, larger adjustments are made. As a result, the outcome moves more towards his opinion, but it may move away from the opinions of other group members. Accordingly, these other group members may adjust their influence weights in the following influence round and the expected outcome moves back. On a larger time-scale, the group outcome could bounce hence and forth on the opinion scale.

We postulate two necessary, but not sufficient conditions, under which *unstable behavior* may arise from a group with initially heterogeneous opinions.

1. The group contains at least two ‘stubborn’ actors. These actors do not change their opinion, but they may adjust their influence weights proportional to their relative frustration. As we showed above, the presence of two stubborn actors is a sufficient condition for non-trivial polarization. In the context of our dynamic weight model, persistent polarization is in turn a necessary condition for the possibility of permanent instability. The reason is that without persistent polarization, the distance between the opinions of the actors will sooner or later approach zero. Notice that the introduction of dynamic weights has not changed this inherent property of the standard model. Even with dynamics weights, the new position that a non-stubborn actor takes after an influence round will always be in between the most extreme positions taken by anyone at the outset of the round. Accordingly, without stubborn actors, the spread of opinions can only decline, regardless whether influence weights are adjusted to expected frustration.

2. There is at least one 'bouncing' actor, i.e. a group member who can be reached by both stubborn actors. This group member is subjected to the changing influence weights of the stubborn actors.

Under one additional condition we expect *persistent instability* to arise in the process of opinion formation within a group containing heterogeneous opinions:

3. The level of intolerance for discrepancy between the expected group opinion and own opinion is sufficiently high, i.e. the intolerance parameter τ should exceed a critical level τ^* in order to induce persistent instability.

We expect that these three conditions also apply to subgroups instead of group members, in the case of large size groups. The minimal group size that meets the necessary – but not sufficient – conditions is three. A one-person group constitutes no group. A two-person group never contains at least two stubborn actors. A three-person group can contain two stubborn actors and one bouncing actor if the weight matrix has a very specific structure: the structure that we displayed in figure 2.

Figures 3a through 3c show how opinion formation with dynamic influence weights develops over time for the 3-person group of figure 2. We used the same initial opinions and the same static weight matrix **Y** as values of the exogenous variables. The only difference between figures 3 and 2 lies in the introduction of the feedback effect of expected frustration. Differences in patterns therefore are strictly the result of endogenous processes. Clearly, figure 3 demonstrates that our extension of the standard model generates non-monotonic changes in opinion occur over time.

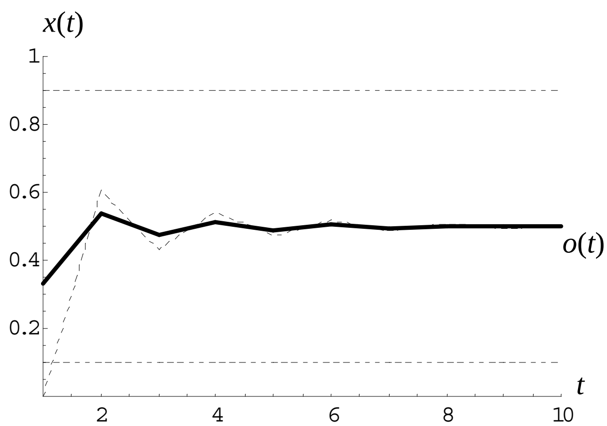

Figure 3a. Unstable convergence of the average opinion $o(t)$ in a *3*-person group (intolerance level τ=2; weight matrix as in figure 2).

Figures 3a and 3b clearly display a pattern of 'unstable convergence.' The outcome $o(t)$ bounces back and forth on the opinion scale, but in due time the process stabilizes towards one single outcome. The difference between figures 3a and 3b is the presumed level of intolerance of the group members: in figure 3b we assumed to be less tolerant for discrepancies between their opinion and the expected outcome. As a consequence, the opinion formation process takes more time to converge. With τ=2, convergence occurs within approximately 10 iterations. With τ=3, the group opinion has still not stabilized after about 50 iterations. Further simulations showed that convergence required about 1500 iterations under τ=3. Note that in both cases the two stubborn actors, with initial opinions $x_1(t_1) = 0.9$ and $x_3(t_3) = 0.1$, firmly maintain their opinions. Only the 'bouncing actor' 2, with his initial position of $x_2(t_2) = 0$, exhibits an unstable behavioral pattern.

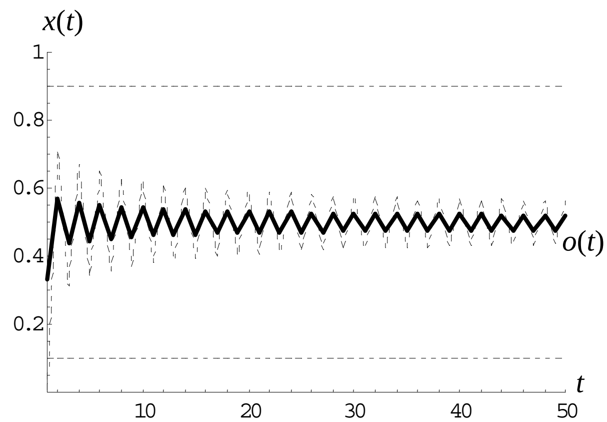

Figure 3b. Unstable convergence of the average opinion in a *3*-person group (intolerance level τ=3; weight matrix as in figure 2).

Figure 3c shows the persistent instability that arises if the intolerance parameter τ is sufficiently large. The 'bouncing' group member 2 oscillates between the opinions of the two 'stubborn' group members 1 and 3. Their opinions $x_1(t_1) = 0.9$ and $x_3(t_3) = 0.1$ have become the amplitudes of a cyclic process that fails to converge.

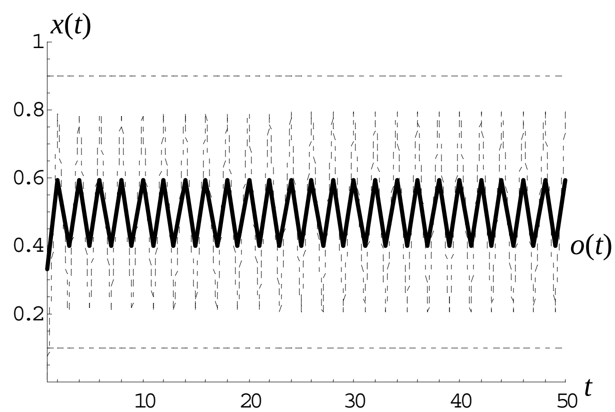

Figure 3c. Persistent instability of the average opinion $o(t)$ in a *3*-person group (intolerance level τ=4; weight matrix as in figure 2).

To test whether this instability is persistent, we replicated the simulation with 10.000 influence rounds instead of 50. We found that even with this large number of influence rounds, changes in group opinion between two subsequent rounds remains identical. Consequently, the pattern displayed in figure 3c is the reflection of genuine and persistent instability. It is extremely unlikely that unstable convergence occurs into one outcome that may be reached when time approaches infinity.

4. Example: complex instability in a six-person group

In the previous section, we showed how a sufficient level of intolerance could generate persistent instability if two stubborn actors and a bouncing actor are present in the influence network. The pattern of instability we found is extremely simple. As figure 3c shows, after some time the dynamics stabilizes into a simple oscillation that repeats itself after two influence rounds. We expect that this pattern is a consequence of the simple structure of the three-person weight matrix rather than a fundamental characteristic of our non-linear model of opinion formation. In this weight matrix, there is only one group member who changes his position. With only one "bouncing actor" present, every position of the group outcome exactly corresponds to one position on the opinion scale that the bouncing actor can take. If, by contrast, a "bouncing subgroup" is present in the weight matrix, one and the same group outcome may be generated by different distributions of opinions in the group. Accordingly, it is not only the previous group outcome that shapes the next influence round, but is also the internal state of the "bouncing group" that matters. Clearly, more complex influence dynamics may result if members of a 'bouncing' subgroup change their opinion than if only one actor does so.

$$x(t=1) = (0.9, 0.3, 0.5, 0.1, 0.4, 0.1)$$

$$v = (0,0,0.6,0,0.4,0)$$

$$Y = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0.3 & 0.15 & 0.15 & 0 & 0 \\ 0.3 & 0.3 & 0.2 & 0 & 0.2 & 0 \\ 0.5 & 0.25 & 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0.25 & 0.25 & 0.25 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

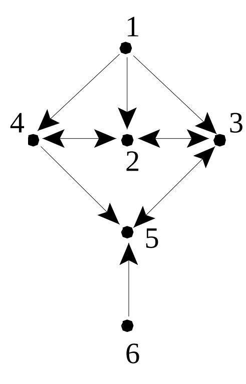

Figure 4. Initial values of the exogenous variables: weight matrix and its network representation, voting power and initial positions in the *6*-actor group.

To study the possible complexities that arise from a bouncing subgroup, we investigated a six-actor group. In the weight matrix two stubborn actors and a bouncing subgroup are defined. The example we study here is derived from a political decision setting as it may typically occur in a small-scale local community. The group is now interpreted as a policy network and its members as corporate actors representing organized interests (Laumann and Knoke 1987). Figure 4 shows the presumed social structure of the group: the static weight matrix **Y** and its network representation, together with the assumed initial values of voting power v and initial positions $x(t=1)$. On the basis of these values we ran a series of simulation analyses.

4.1 Interpretation: A small size policy network

For illustration, the weight matrix in Figure 4 is interpreted as a policy network in a small town. City council (2) faces a decision about imposing environmental regulations on the local plant of a multinational enterprise (1). The international company is one of the two stubborn actors in this process. Because it is extremely powerful it cannot be influenced by any of the other actors in the policy network. The other stubborn actor has opposed interests: a national pro-environmental organization (6), which is likewise insulated from local political pressures. Both stubborn actors have no formal voting power in city council. Voting power over the decision accrues to two local political parties (3) and (5) who form the enacting coalition. Both coalition parties influence each other. Who furthermore influence the political parties? The more conservative party (3), with most voting party, interacts with organized business interests in the city (2) and is influenced by the multinational enterprise. The more liberal party (5) is being influenced by the environmentalists and, additionally, by the mayor of the city (4). In this example, the environmentalists and the multinational enterprise (the stubborn actors) generally exert their vote in this process via indirect influence channels. The multinational enterprise (1) influences the decision indirectly by the mayor and the business interests lobbyists, and directly via its influence channel to the more conservative party. The environmental lobby only influences the smaller, liberal coalition party.

4.2 Complex patterns of persistent instability

We explore the effect of complexity of the ‘bouncing’ subgroup on the instability patterns. To demonstrate the effect of intolerance and on this multi-actor decision process, we replicate our analysis of the 3-person group. We compare the dynamics for three qualitatively different levels of intolerance: *low* intolerance ($\tau = 2$), *moderate* intolerance ($\tau = 3.5$), and *high* intolerance ($\tau = 5$). Figures 5a, 5b, and 5c show simulation results of the first 50 iterations of the corresponding dynamics.

Figure 5a shows that the behavioral pattern under *low intolerance* is qualitatively very similar to the corresponding dynamics of the 3-member group in figure 3a. After a relatively short period of initial instability, opinions stabilize into persistent polarization around approximately round 15 of the influence process.

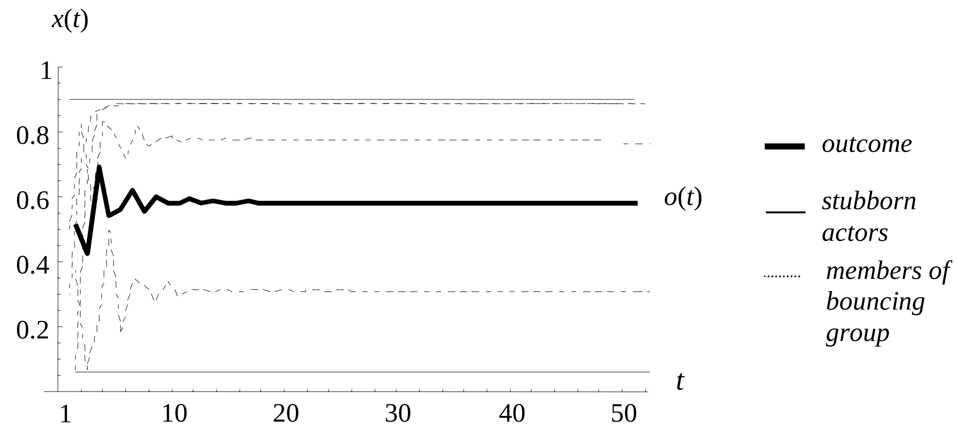

Figure 5a. Unstable polarization in a *6*-actor group with low intolerance level τ =2.

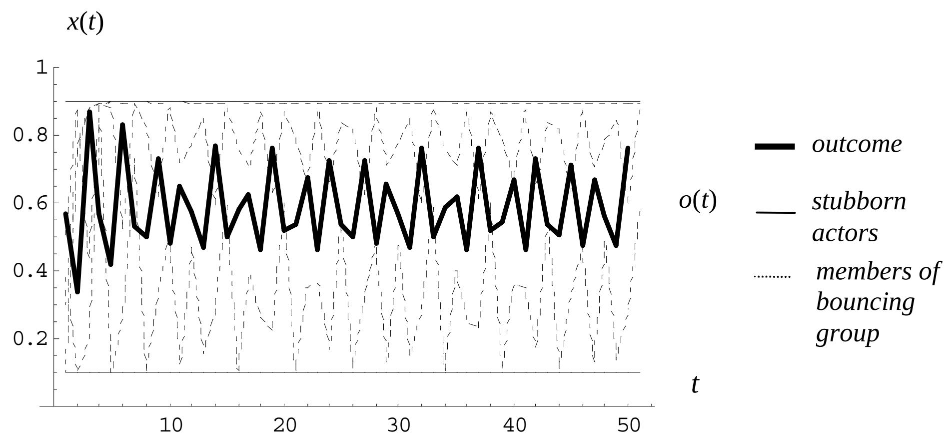

Figure 5b. Complex persistent instability in a *6*-actor group with moderate intolerance level τ =3.5.

Figure 5b shows that with *moderate intolerance* both the three-member group in figure 3b and the six-actor group in figure 5b display persistent instability. In addition, figure 5b reveals that moderate intolerance in our six-actor group entails a qualitatively new form of influence dynamics. Comparison of the figure 3b and 5b shows that the dynamics are considerably more complex in the six-actor group. Two different processes could trigger the pattern of non-linear behavior with moderate intolerance: (a) complex oscillation or (b) deterministic chaos. It is well known that even relatively simple non-linear systems may generate deterministic chaos. The defining characteristic of deterministic chaos is that the trajectory of the system never repeats itself (Schuster 1988) in contrast to an oscillation.

At this point it suffices to note that the dynamics in Figure 5b are persistently instable and exhibit much less repetition in its opinion pattern than does any of the other opinion dynamics we found earlier. Visual inspection of figure 5b shows that after about 10 iterations, the influence dynamics seems to approach a pattern of changing outcomes that repeats itself around iteration 30. A more appropriate analysis is required to establish that this pattern represents an oscillation at all. Visual inspection cannot rule out the possibility that the dynamics of figure 5b exhibit deterministic chaos. Further below, we will inspect in more detail whether the dynamics that arise in the six-actor group may indeed be characterized as a chaotic.

Figure 5c finally shows that there is no simple relation between the level of intolerance and the complexity of behavior. *High intolerance* suppresses the complex oscillation, which arose under moderate intolerance. The influence dynamics narrow down to a simple oscillation between the most extreme positions the bouncing subgroup can possibly adopt.

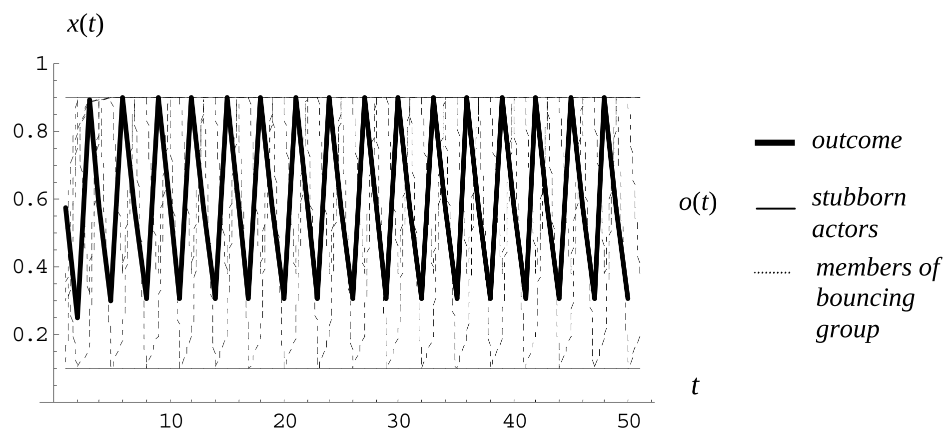

Figure 5c. Simple persistent instability in a *6*-actor group with high intolerance level $\tau = 5$.

How can we explain the transition to a completely different pattern of behavior when the value of intolerance τ increases? The highly intolerant stubborn actors allow members of the bouncing group only a very narrow range of opinions to adopt. Under moderate intolerance, stubborn actors allowed at least some discrepancy between their own position and the decision outcome. As a consequence, members of the bouncing group shifted their positions only to a limited degree towards the positions of the stubborn actors, unless the expected frustration of the stubborn actors became very high. Under high intolerance, the stubborn actors now mobilize so much effort that members of the bouncing group are very soon pulled back and forth between the most extreme positions that they can possibly adopt, given the influence of other group members.

4.3 Exploring the parameter space of τ

Our analysis of the effects of intolerance in the *6*-actor social structure suggests that at least three qualitatively very different types of non-linear behavior occur under different values of intolerance. We even expect phase-transitions to occur: dramatic changes in behavior resulting from only a marginal increase in the level of intolerance. Under low intolerance, opinions rapidly converge to one stable group opinion. Under moderate intolerance, a complex oscillation or even deterministic chaos in the decision outcome over time may arise. Under high intolerance instability reduces to a simple oscillation.

To test this conjecture, we replicated the simulation of the six-actor group with conditions as displayed in figure 4 for 200 influence rounds. In this replication, we varied the level of intolerance, τ, systematically between $\tau = 2$ and $\tau = 5$ in small increments of 0.01. To display the complexity of the dynamics in 200 influence rounds we plotted all group opinions that occur within one simulation after iteration 20 on a horizontal line representing the opinion space [0, ..,1]. Consequently, on each horizontal line 180 dots are plotted. To show how long-run dynamics change with changing values of τ, we chart on the vertical dimension the level of intolerance. Figure 6 shows the results.

Figure 6 shows indeed three qualitatively different levels of intolerance. Below approximately $\tau = 2.8$, intolerance is so low that a stable group opinion emerges.[9] All 180 outcomes are plotted on one position. The graph shows that in this region of the parameter space there is a unique group opinion after iteration 20 at all levels of τ. Above approximately $\tau = 4.5$, intolerance is so high that decision dynamics bounce back forth between only three different outcomes. These outcomes correspond to a cycle of length 3 that passes through: (a) two group opinions maximally close to the positions of the two stubborn actors, and (b) one intermediate outcome in between.

In the intermediate range, between $\tau = 2.8$ and $\tau = 4.5$, figure 6 reveals a complicated dependency of decision dynamics on intolerance. From the value of $\tau = 2.8$ bifurcation arises: two and rapidly more different outcomes obtain. From the overall pattern in this region we draw two observations. Firstly, the group opinion passes through cycles with a large number of different outcomes. For example, at τ=4, the group opinion varies across 9 different opinion levels, ranging between approximately o=0.4 and o=0.8. Secondly, small changes in the level of intolerance may have enormous effects on the influence dynamics. For illustration, a slight increase in intolerance from τ=4 to only τ=4.01 dramatically increases the number of different group opinions from 9 to 21.

[9] More precisely, below $\tau = 2.8$ the dynamics of opinion formation are characterized by unstable polarization, where opinions stabilize into an stable individual opinion within 20 influence rounds.

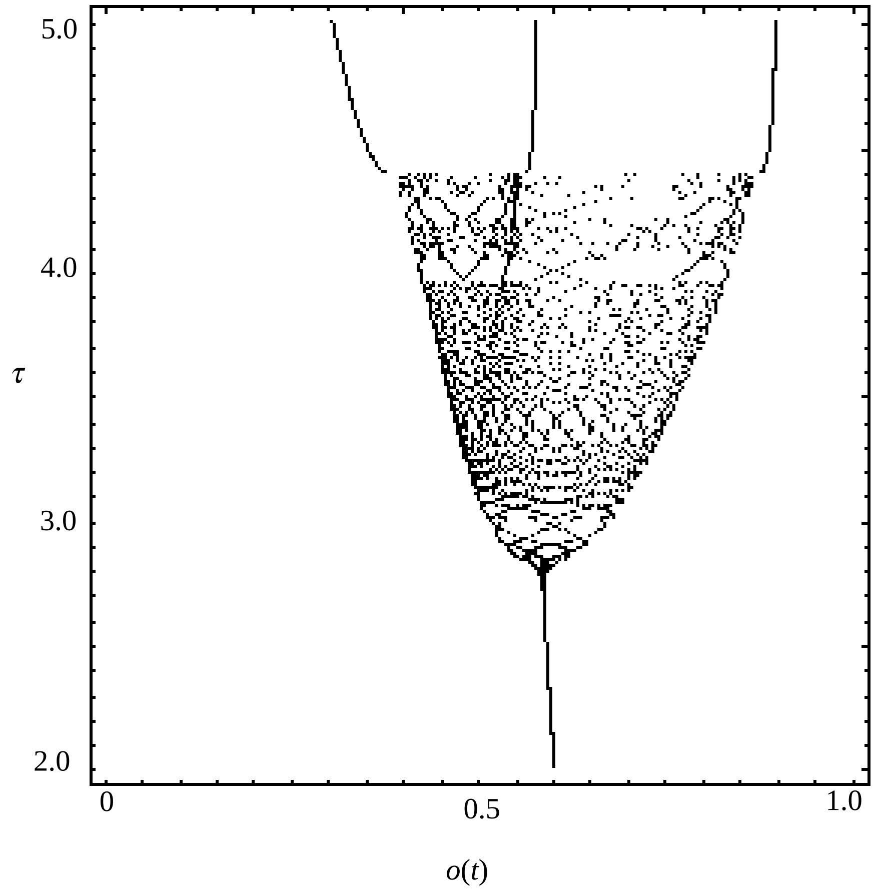

Figure 6. Bifurcation diagram. Effect of intolerance τ on the long-term distribution of group opinion $o(t)$ in the *6*-actor group of figure 4.

4.4 Deterministic chaos or complex oscillation?

The effects of intolerance in the intermediate range suggest the possibility that decision dynamics exhibit properties of deterministic chaos. However, visual inspection is not sufficient to distinguish complex oscillation from deterministic chaos. There are a number of statistical methods that allow a quantitative assessment of the pattern of the dynamics. For our purposes, we apply the method of a Fourier transformation of the time series of group opinions (cf. Schuster 1988). The Fourier Transformation $F(f)$ indicates the degree to which there is an oscillation with frequency f in a time series. If there are a few clear peaks in the Fourier transformation F, this indicates that the time series is generated by overlapping oscillations rather than by real deterministic chaos. By contrast, real deterministic chaos generates an instable pattern of opinions that never repeats any part of itself. Correspondingly, the Fourier transformation of a chaotic time series shows no clear peaks. Instead, every frequency is more or less equally represented in the dataset.

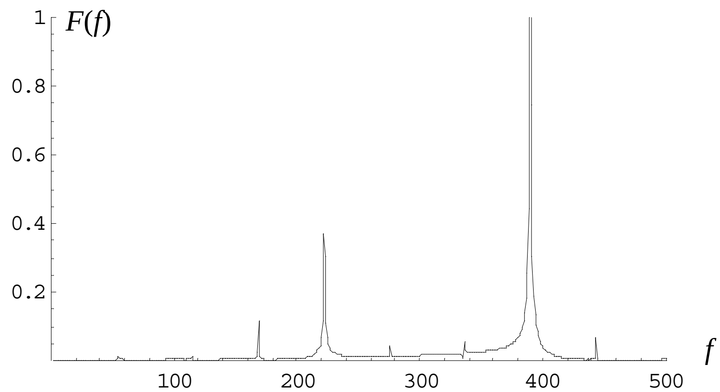

Figure 7. Fourier transformation of the complex cycle of the group opinion $o(t)$ in 6-actor group (intolerance level τ =3.5).

Figure 7 shows the Fourier transformation of a time series of group opinions for one of the cases that are most disordered according the simulations of figure 6, the case of τ=3.5 shown in figure 5b. We ran a simulation with 1000 iterations. The Fourier transformation shows that, despite its complexity, the opinion dynamic is not chaotic. Instead, the time series consists of overlapping oscillations that generate main frequencies of about 390 and 222, corresponding to cycles of length 2.69 and 3.85, respectively. For reliability, we repeated this analysis for a number of other levels of intolerance in the intermediate range of figure 6. This analysis confirmed for all of the intolerance levels we inspected that dynamics consist of complex oscillations rather than deterministic chaos.

5. Conclusion and discussion

The literature on social systems in general and collective decision-making in particular reports the likelihood of unstable, cyclical or even chaotic patterns of behavior. The aim of this paper was to formulate a model of opinion formation / collective decision-making, which: (a) is as simple as necessary to generate non-linear instability; (b) is closely connected to a commonly used model of opinion formation. Accordingly, we obtain the theoretical tools necessary to allow us to identify *conditions* and *mechanisms* under which instability may occur. We analyzed the standard model of opinion formation developed by French (1956) and subsequently extended by other scholars (Harary 1959; Abelson 1964; Friedkin 1991). Our analyses showed that the social influence mechanism in this model produces inherently stable outcomes, with polarization as the most extreme pattern. We considered this property of the standard model unsatisfactory and suggested a new simple and substantive assumption. The assumption we make is that social actors change their influence in order to avoid a future outcome they do not like.

We arrive at two main results. Our first result is that, by making one simple assumption, non-linear behavioral patterns emerge from the model of opinion formation known to be inherently very stable. Within one overarching model of opinion formation and collective decision-making we found three patterns in group opinions and collective decisions. In the first place, stable convergence and stable polarization arise under certain conditions. In the second place, unstable convergence to unanimity and unstable polarization occurred; processes which stabilized only in the long run. In the third place, some patterns reflected simple persistent oscillations. In the fourth place, we found very complex persistent oscillations. We did not find any evidence for the presence of deterministic chaos.

Our second main result is that we arrived at specific conditions under which we expect non-linear outcome patterns in opinion formation and collective decision-making. These conditions are not derived from a textbook model in the literature on non-linear dynamics, but rather from the adaptation of a commonly used social influence model. Consequently, we are able to interpret our results in the context of opinion formation and collective decision-making. Empirically observed instability in these influence processes can be directly related to the appropriate model, instead of being explained from a superficial social translation of a textbook process. This enhances our understanding of the mechanisms and conditions that underlie non-linear behavior in opinion formation and collective decision-making.

5.1 Polarized opinion formation and collective decision-making

Our analyses show that unstable, non-linear behavior in opinion formation and collective decision-making will always depart from a situation of persistent polarization. One necessary condition for persistent instability in the non-linear model corresponds with that producing a situation of non-trivial polarization in the standard model. There is no chance for instability without the situation of non-trivial polarization. With trivial polarization we mean the presence of isolated group members. Non-trivial polarization is caused by the presence of at least two 'stubborn actors,' able to influence other group members but unable or unwilling to be influenced. However, the existence of polarizing group or community is not sufficient for instability. An additional necessary condition is the presence of at least one 'bouncing' actor, directly or indirectly influenced by both stubborn actors.

From these conditions we infer that the social structure of the influence process, described by the weight matrix **Y** (and interpreted as a social network), crucially determines the likelihood of persistent instability. Whether these types of social networks can be found in reality remains an important, but open question at the moment. In any case, the model has generated specific, testable hypotheses concerning the social structure of decision-making where unstable behavior is reported. Because unstable, non-linear behavior is to be expected when situations of polarization meet additional conditions, the empirical abundance of polarization firmly establishes an upper boundary on the likelihood of persistent instability.

Our adaptation of the standard model of opinion formation has brought about a third condition for non-linearity in opinion formation and collective decision-making. Actors must be sufficiently intolerant for a discrepancy between the expected group opinion and their own opinion, in order to induce persistent instability. For a simple group, with only one bouncing actor, the stubborn actors have a high impact on the behavior of the bouncing actor. For low values of intolerance, unstable polarization occurs. For high values of intolerance, a simple, cyclic pattern emerges. The bouncing group member is not able to stabilize his opinion, because the stubborn actors alternate their experience of the discrepancy. The group is not able to stop the process, because no compromise between the two 'amplitude' outcomes can be obtained.

5.2 Intolerance and unpredictability

When the bouncing actor is replaced by a bouncing subgroup of actors, an intermediate range of values for the level of intolerance comes into existence. Again, the social structure of the influence process makes an enormous difference: the conditionality of the level of intolerance is more precisely defined. Within the intermediate range the group outcome displays non-linear patterns over time. These non-linear patterns are best characterized by highly complex oscillations. The oscillations are complex because: (a) many different outcomes are reached, and (b) periodicity of the oscillations is large: there are considerable time lags between the same outcomes.

The complex patterns do not exhibit the characteristics of deterministic chaos. This result is in full accordance with the results of analyses of economic time series data (Day 1992). Yet, one does not need chaos theory to arrive at complex instability and problems of prediction. The patterns of group outcomes we generated in the intermediate range of intolerance τ still have important implications for the predictability of group opinions and collective decisions. If the value of τ is within this range the following arguments hold. The large periodicity as well as the multitude of possible group outcomes makes it extremely difficult for social actors – as well as analysts – to foresee the outcomes of opinion formation or decision-making. The outcomes of influence processes become sheer unpredictable for two reasons. First, the value of the group outcome is highly dependent upon the time-step at which a collective decision is made. An unforeseen variation of only one time-step can make a tremendous difference in the value of the predicted outcome. Secondly, we have seen that the range of possible outcomes is highly sensitive to very small changes in the value of intolerance parameter τ. If polarized opinion formation or decision-making meets the conditions to turn into persistent complex instability, then a correct prediction requires the extremely precise estimation of the value of τ. It is highly unlikely that such precision can be obtained in practice.

As in the case of the social structural conditions, we must turn to the empirical validity of the assumption of intolerance for discrepancies between opinions and expected outcome. As such, the assumption seems to be plausible, but how likely is an occurrence of 'intermediate' intolerance, or 'high' intolerance. The space of the theoretical parameter has no direct relation with empirical situations (one might postulate that in times of severe crisis, time pressures and colossal consequences of incorrect actions could force actors to become intolerant).

Moreover, the instability and non-linear dynamics we generated in polarized opinion formation and decision-making could very well depend upon our assumptions of rationality. Our assumptions are not at odds with current models of learning and optimization (Fudenberg and Levine 1998). The model assumes that knowledge of social actors about the influence structure is incomplete or at most preliminary. For this reason, actors engage in boundedly rational optimization on the basis of the present situation. After each influence round, actors receive new information about the expected outcome and they adapt their influence strategies on the basis of their current knowledge. In other words: social actors are assumed to be relatively blind in past and future. If actors were more rational, they would be able to foresee future unstable behavior. This could provide them with an incentive to reach a more stable agreement. If actors were more capable of learning from the past, they would compare their opinion with an average of the group outcomes in a number of past influence rounds. Intuitively, we would postulate that this mechanism of learning from the past would immediately stabilize the influence process. Future research might show whether granting more intelligence and a memory to social actors in our model warrants even more stability in opinion formation and collective decision-making than we already found.

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