Equilibrium in a Model of Endogenous Political Party Formation*

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Abstract

We develop a model of endogenous party formation in a multidimensional policy space. Party platforms depend on the composition of the parties' primary electorate. The overall social outcome is taken to be a weighted average of party platforms and individuals vote strategically. Equilibrium is defined to obtain when no group of voters can shift the social outcome in its favor by deviating and the party

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platforms are consistent with their electorate. We provide sufficient conditions for existence and study the properties of sorting equilibria in this model.

1 Introduction

In the extensive literature on political economy and party competition it is commonly assumed that there exists a policy space in which the platforms proposed by the parties can be seen as points. Voters have well defined preferences over such a space and, depending on the type of electoral rule, might or might not vote in a strategic manner. Under the classical Downsian approach the objective of the parties is assumed to be wining the election. In equilibrium each party announces a platform that maximizes the number of votes in its favor, taking the other parties' platforms as given. Many recent papers, however, assume that parties are "ideological", that is, that they have preferences over the policy space. In the latter interpretation one may view parties as institutions that represent contesting interest groups in the society. An ideological party adopts a platform that maximizes its utility, subject to electability considerations. This approach was introduced by Donald Wittman [23] and has been extensively developed in later years. A recent and very complete version of such a model is provided in Roemer [22] (see also Osborne [18] for a survey that covers both the Downs and the Wittman approach). Most of the papers in the Wittman tradition share the same basic assumption: parties, and their ideology, are exogenously given.

In this paper, we provide a theory of endogenous party formation in a multidimensional policy space. Agents are seen in a twofold role, both as voters and as party members. Each agent can belong to (at most) one party. The preferences (ideology) of each party are determined, according to some aggregation rule, by the preferences of its members. The preferences of the parties will determine the electoral platforms. After both parties announce their platforms, agents vote, possibly in a strategic way. It may, however, happen that some agents prefer to vote for a party different from the one they belong to. We view such a situation as unstable and inconsistent with the system being in the overall equilibrium. Indeed, in that case it is sensible to assume that some of those agents will want to change their party affiliation. But this, in turn, will imply a change in the preferences of the parties and, as a consequence, a change in their platforms. Thus, an overall equilibrium requires that it is in the interest of each agent to vote for the proposal of his or her party. In this way, the membership of each party is endogenously obtained as a function of the distribution of voters' preferences. The basic idea here is the same as the one used by Baron [6], Ortuño–Ortín and Roemer [17] and Roemer [22], and it is inspired by the "voting with one's feet" models (see Caplin and Nalebuff [9]). A similar strand of literature, which studies political party activism with endogenous political party platforms, is represented by Aldrich [1] and Poutvaara [20]. D. P. Baron [6] considers a multidimensional model with strategic party behavior and sincere voting. Existence of equilibrium, however, is only provided for a very specific two–dimensional example, with three parties and a uniform distribution of voters. I. Ortuño–Ortín and J. E. Roemer [17] consider a specific example of endogenous party formation in which the policy space is one–dimensional. J. E. Roemer [22] deals with a two–dimensional policy space problem but the nature of the political parties and the equilibrium concept are different from those standard in the literature.

Thus, our model differs from [17] and [22] in several important respects. First, we provide general existence results, whereas [6] and [22] provide only a numerical example in a two–dimensional model. Second, and more important, we don't restrict ourselves to the case of a one–dimensional policy space, unlike the example provided in [17]. In fact, proving existence of equilibrium in the one–dimensional case is relatively easy. For higher dimensions, however, a different approach to establishing equilibrium existence is needed. While the resultant proof is quite involved, it provides important new insights in the nature of party formation. Thus, one message of the paper is that the odd–even dimensionality of the policy space might play an important role to understand the formation of parties. This apparently paradoxical result recalls the one established by Caplin and Nalebuff [9] and explored in Gomberg [12].

An important feature of the paper is the way in which the outcome of the electoral competition is modeled. We assume that the implemented platform does not need to coincide with the platform of the winning party. Namely, the implemented policy is taken to be some convex combination of the parties' platforms. The weights in this combination are assumed to be an increasing function in the vote share (see Grossman and Helpman [13], Ortuño–Ortín [16], Gerber and Ortuño–Ortín [11], Alesina and Rosenthal [3, 4]). We believe that this is a realistic assumption, which captures the way many democratic societies adopt policies.

In a related approach, Osborne and Slivinski [19] and Besley and Coate [7] provide a model of "citizen–candidates" in which candidates are also endogenously determined. In those models, candidates, which play a role similar to parties in our setting, cannot commit to policies. Consequently, the proposals, or platforms, must coincide with the ideal policies of the parties (see also Alesina [2] and Alesina and Rosenthal [3] for this assumption). Unlike our model, this literature takes the candidates to exist independently, outside any political party structure. Another important difference concerns our analysis of coalition formation by voters. In our model, voters can form such coalitions; consequently, in equilibrium profitable deviations by coalitions are not allowed (a similar assumption also appears in Alesina and Rosenthal [3, 4] and Gerber and Ortuño–Ortín [11]).

To sum–up, this paper provides an endogenous theory of party formation in a multidimensional setting with the following features:

i) Parties do not necessarily behave strategically in their choice of platforms.

ii) The implemented policy is a convex combination of all the platforms.

iii) Agents vote in a strategic way and are allowed to form coalitions.

Note that imposing that parties always behave strategically in their choice of platforms would generate serious difficulties in the model. The reason is the well known existence problem of (Nash) equilibrium in political games when the policy space is multidimensional[1]. This problem disappears, for example, when parties are not able to make commitments so that voters associate each party with its ideal policy. Our approach will be general enough to encompass strategic behavior, whenever it is well defined. As well as other types of behavior, like parties proposing their ideal policies. It is not our aim to model explicitly the way in which parties choose their platforms. We only impose certain natural restrictions on their choices.

It is important to notice that among these three main features of the model only the first one is essential for our results. In particular, one can easily extend our model for the case in which agents vote sincerely, i.e. agents vote for the platform they like the best according to their preferences, and the implemented policy coincides with the platform of the winning party[2].

[1] For a general view of the problem see, for example, Roemer [22]. See also Roemer [21] for existence of an alternative equilibrium concept in the two–dimensional case.

[2] In this case, it is possible to introduce without any difficulty uncertainty in the model. For example, the outcome of the election could be uncertain. The results here wouldn't differ from the ones in the certainty case.

Thus, our method can be readily modified into a "winner takes all" model and the same results will hold.

The organization of the rest of the paper is as follows. Section 2 sets up the basic model. Section 3 discusses the outcomes of voting games obtained by fixing party platforms. Section 4 provides a two–party equilibrium existence result for the case when the number of policy dimensions is odd. Section 5 analyzes the case of two policy dimensions under the additional assumption that parties choose policies by a mean voter rule. Section 6 discusses the robustness of such equilibria to small changes in the specification of the model. Section 7 concludes.

2 The model

Consider a society consisting of a continuum of heterogeneous individuals, with the set of possible types denoted by $A \subset R^n$. The set of agents shall be represented by a measure space $(A, \mathcal{B}, F)$, where $\mathcal{B}$ is the σ–algebra of (Borel) subsets of A and F is a measure over the type space. We shall assume that F is finite (and, therefore, we may normalize $\int_A dF = 1$), with compact support and hyperdiffuse (that is, every $(n-1)$ dimensional hyperplane in A is of zero measure).

As it is standard in the theory of non–atomic games, measurable subsets $B \in \mathcal{B}$ of the type space A shall be called **coalitions**. If for some coalition $C \in \mathcal{B}$, $F(C) = 0$, it shall be called a **null** coalition.

Let there be a fixed number of political parties[3] $M = \{1, 2, ...m\}$. Individuals are free to join any of the parties, resulting in a population partition. Strictly speaking, a partition is a collection of measures $\{F_j\}_{j \in M}$ over A such that for any $E \in \mathcal{B}$, one has $\sum_{j=1}^{m} F_j(E) = F(E)$. However, we shall soon impose assumptions which will insure that individuals of the same type will always be strictly better off by going to the same community (except for, possibly, a null coalition of agents), and consequently the F_j's are mutually singular. This suggests that it may be convenient to restrict our attention to population partitions $C = \{C_j\}_{j=1}^{m}$, with $C_j \in \mathcal{B}$, that are also partitions of the type space A into m coalitions. The set of all such partitions we shall denote as Σ. Given such a partition $C \in \Sigma$, the membership share of the

[3] The set up of the model allows for more than two parties. Our results, however, will be proven for the two–party case.

party $j \in M$ is $w^j = w^j(C_j) = F(C_j)$. Hence, the vector of party weights $w = (w^1, \ldots, w^m)$ is an element of the $m-1$–dimensional simplex Δ^{m-1}.

The society has to implement a vector of policies $x \in X$, where X is a non–empty compact and convex subset of R^n. Every individual of type $\alpha \in A$ cares only about *overall* policy outcomes. For simplicity, we shall restrict the class of individual preferences considered [4].

Assumption A1 (Euclidean preferences)**:** The individual preferences over X of each agent of type $\alpha \in A$ may be represented by the utility function $u(x;\alpha) = -||x - \alpha||$, where $|| \cdot ||$ stands for the Euclidean norm.

Furthermore, we shall assume that X is sufficiently "large" in the sense that every individual's ideal point is part of the set X of feasible policies. In fact, identifying individuals with their ideal policies, we shall assume:

Assumption A2: $K \subset X$, where K is the convex hull of the support of F.

2.1 Policy Outcomes and Voting

Suppose each party $j \in M$ chooses to advocate the policy $p^j \in X$. Facing a policy profile $p = \{p^j\}_{j \in M} \in \prod_{j=1}^{m} X \equiv X^m$, individuals shall vote, in a manner explained below, inducing some population partition C with corresponding vote shares represented by a vector $w(C) \in \Delta^{m-1}$ (when the underlying partition is clear we shall just write w).

The overall policy outcome is a function of the manner in which the vote divides between the parties, as well as of which propositions are on offer. In other words, there is some outcome function $T : X^m \times \Delta^{m-1} \to X$. While, in principle, a general set of outcome functions may be analyzed, we may want to restrict ourselves to special classes of these. In particular, in this paper we shall focus on the "convex combination" (or "weighted average") outcome functions.

Assumption O1: $T(p, w) = \sum_{j=1}^{m} g^j(w)\, p^j$, where $g \equiv (g^1, ...g^m)$ is a continuous function from Δ^{m-1} to itself.

[4] An extension of the model to accommodate linear preferences, $u(x;\alpha) = \alpha \cdot x$, as in Gomberg [12] is, in fact, straightforward.

This assumption entails that the actual policy implemented in the society is a consequence of a political compromise between the competing parties. Of course, we will assume below that the weight each party has in the final outcome is directly linked with the support it obtains.

Notice that for $\varepsilon > 0$, arbitrarily small, we can make the function g take the values $g^j(w) = 1$, for $w^j > 1/2 + \varepsilon$. In other words, by choosing the function g appropriately, one can approximate arbitrarily the "winner takes all" situation, i.e., $g^j(w) = 1$, for $w^j > 1/2$. At any rate, as already mentioned in the introduction, with minor modifications our results can be directly established as well (without having to deal with approximations) for the "winner takes all" case.

The following monotonicity assumption on the outcome function shall be imposed throughout.

Assumption O2: (strict monotonicity) For every $i \in M$, the weight $g^i(w)$ assigned to party i is strictly increasing in the vote share w^i obtained by that party.

Finally, we assume that, if a party policy proposal attracts no voters, it should have no weight in the final outcome.

Assumption O3: (irrelevance of null voter coalitions) For every party $i \in M$, we have that $g^i(w) = 0$ whenever $w^i = 0$.

It follows from O3 that, for each $i \in M$, if $w^i = 1$, then $g^i(w) = 1$. Given a policy profile $p \in X^m$ and the outcome function we define the voting game as follows. Each individual votes for one of the political parties. Under the assumption of Euclidean preferences, given the policy proposals represented by p and the voting pattern represented by w, the payoff enjoyed by an individual of type α is given by $u(T(p, w); \alpha) = -||\alpha - T(p, w)||$.

As it is standard in the multi–jurisdictional literature, we have postulated that there is a continuum of voting agents. This is done in order to avoid existence problems resulting from the non–convexity of the individual choice set. However, the continuum assumption has its costs as well and it introduces some technical problems.

In particular, in the context of a model with a continuum of voters, we face the usual problem of voting incentives: since no individual by himself impacts the outcome, any voting behavior may be rationalized. We could have assumed that agents vote sincerely for the party whose policy platform they like the most. However, we are interested in studying implications of some sort of strategic behavior on behalf of the voters[5]. This requires to use one of the equilibrium refinement concepts based on the possibility of deviation by (non–negligible) coalitions. The voting equilibrium concept we employ here is, essentially, the Aumann [5] Strong Nash Equilibrium (SNE), modified to accommodate the model with a continuum of agents.[6]

In general, the problem of existence of SNE in a voting game like the one defined here, is highly non–trivial. Furthermore, even if an SNE exists, it may not be unique. However, for the two–party case it may be shown that, for any policy proposal profile with parties taking distinct policy positions, there does indeed exist a unique SNE of the voting game.

In order to minimize the amount of notation involved it is convenient to express strategy profiles of voters in terms of the partitions of the type space A among parties.

Definition 1 *A voting partition* $C = \{C_j\}_{j=1}^m \in \Sigma$ *shall be called a Strong Nash Equilibrium (SNE) of the voting game given by the policy proposal profile* $p \in X^m$*, if there does not exist a partition* $C' = \{C'_j\}_{j=1}^m \in \Sigma$ *such that for all individuals of types* $\alpha \in B = \bigcup_{j=1}^m C_j \cap (A \backslash C'_j) \in \mathcal{B}$*,*

$$u(T(p, w(C')); \alpha) \geq u(T(p, w(C)); \alpha)$$

and, the set of agents $\alpha \in B$ *for which the above inequality is strict has strictly positive measure.*

2.2 Party policy choice

So far, we have essentially assumed that the policies staked out by the parties are exogenously given. However, party positions naturally depend on their

[5] The main results of the paper also hold in the case agents vote sincerely for the party whose policy platform they like the most.

[6] SNE may seem to involve much more coalitional reasoning than is likely in a model with a continuum of agents. An alternative would be to consider deviations only by small (but non–null) coalitions of "alike" agents. This could be justified by claiming that similar agents are more likely to be able to communicate in order to be able to behave as a coalition, or by having agents believe that their behavior provides an "example" followed by other individuals with similar preferences. In terms of this model, however, employing this, in principle, less restrictive equilibrium concept has the same effect as employing the SNE.

constituents. We assume that each party possesses a **statute**, which may be viewed as a mechanism for establishing a policy platform, or **program,** as a function of the environment. As it has just been discussed, a platform may be viewed as a policy vector $p^j \in X$, which the party $j \in M$ would implement, if it could single–handedly determine the society's policies. It will be generally assumed that a party's platform is always well defined, given some set of relevant data. If party $j \in M$ takes into account just the way in which the population is partitioned, its statute may be viewed as a function $P^j : \Sigma \to X$. Such a party shall be called **membership–based**.[7] If all parties are membership based we denote the profile of statutes as $P : \Sigma \to X^m$.

Although $P(C)$ takes as an argument the partition of the entire population, the decisions of some people may actually be irrelevant for the party policy choice. In fact, our model does not require that every citizen joins a political party. In this case, we may want to view the overall party membership as the set of "political activists" (possibly, more radical, or just more interested in party politics), in the spirit of Aldrich [1].

A typical example of a membership–based rule would be the median–voter rule, which tells each party to choose the ideal policy of its median member. While, this rule is only defined when individuals vary in a single dimension, in a multi–dimensional context we may study, for example, the mean–voter rule.[8] In general, any profile of social choice rules, aggregating preferences of the members of each party would be in this class.

Notice that parties do not necessarily choose their policy platforms in a strategic way. This is reasonable if, for example, we see P as the function determining the ideal policies of the parties and they cannot make credible commitments to policies. Under this interpretation, once parties are formed, agents can observe their membership and infer the ideal policy that each party will try to implement. Thus, voters will not believe announcements

[7] In principle, more general domains for party statutes may be assumed. In particular, it may be of interest to study the case in which parties respond in some sense to others' actions. An extension of the model along these lines is actually quite straightforward, as suggested by Caplin and Nalebuff [9].

[8] This seems particularly fitting in the present context, since, under appropriate conditions, this may be interpreted as the policy proposal that cannot be defeated by a super–majority of party's members in a binary voting (Caplin and Nalebuff [8]). See also Aldrich [1] and Baron [6], where it is assumed that either the party platform or the ideal policy coincides with the mean ideal policy of a party activist.

different from P (see Alesina [2])

2.3 Equilibrium

In much of the earlier literature, the internal and external politics of the parties have been treated separately. Nevertheless, the two are obviously interrelated, in the sense that party membership determines policy platforms and policy platforms serve to attract citizens to parties. Assuming that party membership coincides with party electorate (an assumption that may be relaxed along the lines discussed in Section 2.2), we say that equilibrium obtains when the voting partition resulting from a policy profile on offer coincides with the membership partition inducing this policy profile.

It is easy to construct equilibria with parties being identical in their policy positions. In this case, any voting pattern corresponds trivially to an SNE of the voting game. Therefore, we are free to chose a population partition to support the identical party positions. Given the apparent "pluralism" of positions on policy issues observed in most political systems, it is, however, of interest to study existence of equilibria with non–identical parties.

Definition 2 *Given an outcome function T and a party policy function P, we say that $(p^*, C^*) \in X^m \times \Sigma$ is a **multi–party equilibrium** if:*
(i) $p^* = P(C^*)$
(ii) C^* is a SNE of the voting game induced by p^*.
*If, furthermore, the equilibrium party proposals are distinct (i.e., $p^{j*} \neq p^{k*}$ whenever $j, k \in M$ and $j \neq k$) such equilibrium is called **pluralistic.***

Given a multi–party equilibrium (p^*, C^*), the associated policy outcome is $T^* = T(p^*, w(C^*))$.

3 Existence of Voting Equilibrium

Before tackling the problem of the pluralistic multi–party equilibrium existence, we first have to find conditions which will ensure that the outcome of the voting game obtained when party platforms are fixed is well defined. In fact, it turns out that assumptions A1, O1 and O2 are sufficient to guarantee existence and uniqueness of SNE in a two–party voting game for a large class of policy profiles. Let $S^{n-1} = \{x = (x_1, \ldots, x_n) \in R^n : \sum_{i=1}^{n} x_i^2 = 1\}$ denote the $(n-1)$–dimensional unit sphere in R^n. From now on, we will restrict our attention to the two-party case.

Proposition 1 *Let $m = 2$ and assume A1, O1 and O2. Then, for each policy profile $p = (p_1, p_2) \in X^2$ such that $p^1 \neq p^2$, the following hold.*

(i) There exists a unique (up to a null coalition of agents) SNE of the voting game;

(ii) In an equilibrium $C = \{C_1, C_2\}$, the voters of each party (except, possibly, a zero measure of them) may be separated by a hyperplane through the set of types A, i.e. $C_1 = \{\alpha \in A : \alpha \cdot \pi \geq b\}$ for some $(\pi, b) \in S^{n-1} \times R$. Furthermore, $T(P(C), C$ is in the hyperplane $\{\alpha \in A : \alpha \cdot \pi = b\}$, separating C_1 and C_2.

$>$From now on, we include, for convenience, the separating hyperplane $\{\alpha \in A : \alpha \cdot \pi = b\}$ of indifferent individuals in C_1. To prove the proposition we shall rely on the following Lemma.

Lemma 2 *Under the assumptions of Proposition 1, a partition $C \in \Sigma$ is a Strong Nash Equilibrium of the voting game if and only if*

$$F\left(C_j \cap \{\alpha \in A : \alpha \cdot (p^j - p^i) < T(p, w(C)) \cdot (p^j - p^i)\}\right) = 0$$

for any $i, j \in M$ such that $i \neq j$.

Proof of Lemma 2. I. Sufficiency. Fix $i, j \in M$ with $i \neq j$. Let $C \in \Sigma$ be a partition such that the above is true. For each $\alpha \in A$, consider $W_\alpha : [0, 1] \to R$ defined by

$$W_\alpha(\lambda) = -||\alpha - \left(\lambda p^j + (1-\lambda)\right) p^i)||$$

Since,

$$\frac{dW_\alpha(\lambda)}{d\lambda} = \frac{1}{W_\alpha(\lambda)} \left(\alpha - \left(\lambda p^j + (1-\lambda\right) p^i)\right) \cdot \left(p^j - p^i\right),$$

the function $W_\alpha(\lambda)$ is increasing (resp. decreasing) in λ whenever $\alpha \cdot (p^j - p^i) - (\lambda p^j + (1-\lambda) p^i) \cdot (p^j - p^i)$ is positive (resp. negative). Therefore, by assumption O2, every agent $\alpha \in A$ such that $\alpha \cdot (p^j - p^i) > T(p, w(C)) \cdot (p^j - p^i)$ will be strictly worse off in joining any (positive measure) coalition of agents deciding to vote for i, and, likewise, every agent $\beta \in A$ such that $\beta \cdot (p^j - p^i) <$

$T(p, w(C)) \cdot (p^j - p^i)$ will be strictly worse off in joining any (non–null) coalition of agents deciding to vote for j.

II. Necessity. Suppose not. Let C be a SNE for which the condition does not hold. Fix again $i, j \in M$ with $i \neq j$ and define

$$D^j = C_j \cap \{\alpha \in A : \alpha \cdot (p^j - p^i) < T(p, w(C)) \cdot (p^j - p^i)\}$$

Clearly, $D^j \in \mathcal{B}$ and, by assumption, $F(D^j) > 0$, for some $j = 1, 2$. Let us suppose, for example, that $F(D^1) > 0$. Since F is hyperdiffuse, there exists $\eta > 0$ such that $F\left(D^1_\eta\right) > 0$ where the set D^1_η is defined by

$$D^1_\eta = D^1 \cap \left\{\alpha \in A : \alpha \cdot (p^1 - p^2) + \eta < T(p, w(C)) \cdot (p^1 - p^2)\right\}.$$

Using hyperdiffuseness of F again, it is easy to show that there is a coalition $D' \subset D^1_\eta$ of "sufficiently small" measure $\varepsilon = F(D') > 0$. Consider the new coalition $C'_1 = C_1 \setminus D'$, $C'_2 = C_2 \cup D'$. That is, in the new partition C', agents in D' have changed their vote from party 1 to party 2.

Since, g is increasing, we have that $w_1(C') < w_1(C)$ as long as $\varepsilon > 0$, so $T(p, w(C')) \cdot (p^1 - p^2) < T(p, w(C)) \cdot (p^1 - p^2)$. By taking ε small enough, and taking into account that g is also continuous, we may also guarantee that $T(p, w(C)) \cdot (p^1 - p^2) - \eta < T(p, w(C')) \cdot (p^1 - p^2)$ as well. But now we see that the members of D' are strictly better off, so C could not be a SNE.

Finally, note that, since F is hyperdiffuse, the partitioning hyperplane of "indifferent" voters

$$H = \{\alpha \in A : \alpha \cdot (p^1 - p^2) = T(p, w(C)) \cdot (p^1 - p^2)\}$$

has zero measure. □

Proof of Proposition 1. Fix the policy proposal $p = (p_1, p_2)$ and, for each $t \in R$, let

$$H(t) = \{\alpha \in A : \alpha \cdot \left(p^1 - p^2\right) = t\}$$

denote a hyperplane, normal to the difference of policy vectors. For each $\lambda \in [0, 1]$ define

$$t^\lambda = \left(\lambda p^1 + (1 - \lambda) p^2\right) \cdot \left(p^1 - p^2\right).$$

Notice that $t^0 = p^2 \cdot (p^1 - p^2)$ and $t^1 = p^1 \cdot (p^1 - p^2)$. Thus, $H(t^0)$ (respectively $H(t^1)$) corresponds to a hyperplane through p^2 (respectively

p^1). Furthermore, since t^λ is strictly increasing in λ, we have that $t^0 < t^1$, whenever $p^1 \neq p^2$.

Let $C(t) \in \Sigma$ denote the population partition induced by $H(t)$, i.e. $C_1(t) = \{\alpha \in A : \alpha \cdot (p^1 - p^2) \geq t\}$, $C_2(t) = \{\alpha \in A : \alpha \cdot (p^1 - p^2) < t\}$. Define the (continuous) map $h : R \to R$ by

$$h(t) = h(t; p) = T(p, w(C(t))) \cdot (p^1 - p^2).$$

For every $t \in R$, we have that $T(p, w(C(t)))$ belongs to the segment $[p^1, p^2]$, because $g^1 \in [0, 1]$. Hence, $t^0 \leq T(p, w(C(t))) \cdot (p^1 - p^2) \leq t^1$. Thus, $h : R \to [t^0, t^1]$. Since, g^1 is strictly increasing, the map h is non–increasing in t.[9] By the intermediate value theorem, the restriction $h : [t^0, t^1] \to [t^0, t^1]$ has a fixed point $t^* \in [t^0, t^1]$, which satisfies

$$T(p, w(C(t^*))) \cdot (p^1 - p^2) = t^*$$

The fixed point t^* is unique because h is non–increasing. By lemma 2, $C(t^*)$ corresponds to a unique (up to a zero measure of voters) SNE of this voting game. □

Besides insuring existence and uniqueness of the SNE in the party voting game, Proposition 1 restricts the set of the population partitions that may emerge as a voting outcome. In fact, if we ignore deviations by null coalitions, then, given any two distinct policy proposals, the population should be partitioned by a hyperplane.

We shall denote the set of all population partitions into two non–empty communities that may be induced by a pluralistic policy profile as $\hat{\Sigma}$. Therefore, in the two–party case $\hat{\Sigma}$ may be taken to be simply the set of all partitions of A by a hyperplane. Each such partition $C \in \hat{\Sigma}$ may be parametrized by the unit normal vector to the partition hyperplane $\pi^C \in S^{n-1}$ (pointing in the direction of C_1) and an intercept $b^C \in R$. As it has been already noted in Caplin and Nalebuff [9], under such a parametrization and ignoring null coalitions, $\hat{\Sigma}$ is identified with an open subset of $S^{n-1} \times R$ which is homeomorphic to the whole space $S^{n-1} \times R$.[10]

[9] It may actually be constant only if shifting the hyperplane implies a change of decision by a null measure of voters, i.e. if the hyperplane $H(t)$ does not intersect the support of F.

[10] As described, $\hat{\Sigma}$ may be identified either with the direct product $S^{n-1} \times R$ (which we may view as a cylinder) or with a Mőbius band. The former, however, turns out to be the case, as long as we care about the orientation of the normal vector π – i.e., the identity of a party adhering to a particular position – which we obviously do here.

Thus, from now on, we will identify the set of population partitions $\hat{\Sigma}$ with the cylinder $S^{n-1} \times R$. And, abusing the notation, when there is no confusion, we will not distinguish between population partitions in $\hat{\Sigma}$ and their associated hyperplanes.

In view of the preceding discussion, the voting behavior may be used to define a mapping from the set of all possible pluralistic policy profiles into the set of population partitions $V : \hat{X}^2 \to \hat{\Sigma}$, where $\hat{X}^2 = \{p \in X^2 : p^1 \neq p^2\}$, as follows.

For each $p \equiv (p^1, p^2) \in \hat{X}^2$, we define the induced orientation vector

$$\upsilon_1(p) = \frac{p^1 - p^2}{||p^1 - p^2||}$$

and let $\upsilon_2(p) \in [t^0(p), t^1(p)] \subset R$ be the unique fixed point of $h(t)$ constructed in the proof of Proposition 1. We define then $V(p) = (\upsilon_1(p), \upsilon_2(p)) \in S^{n-1} \times [t^0(p), t^1(p)]$. Note that other hyperplanes may induce equivalent partitions (i.e. partitions which differ by a null set of agents) as well, as long as the total mass of the population "between" them and the plane $V(p)$ is null.

Lemma 3 *Under assumptions A1, O1 and O2, the function V is continuous.*

Proof: It is immediate that $\upsilon_1(p)$ is a continuous function on $\hat{X}^2$. To see that $\upsilon_2(p)$ is a continuous function, notice, firstly, that $h(t;p)$ is continuous in both variables. Recall, that for each $p \in \hat{X}^2$ there exists a unique fixed point, $t^*(p)$, of $h(t;p)$. This defines a mapping $p \mapsto t^*(p)$ from $\hat{X}^2$ to R. By the Lefschetz's fixed point theorem (see McLennan [14]), for any open neighborhood U of $t^*(p)$ there exists an open neighborhood $W \subset \hat{X}^2$ of p, such that for each $p' \in W$, the unique fixed point $t^*(p')$ must be in U. Hence, $\upsilon_2(p)$ is continuous. □

4 Existence of Pluralistic Equilibrium

In general, the problem of existence of pluralistic equilibria is highly non–trivial. In a two–party case, however, we shall provide a rather strong existence result, albeit depending somewhat on the dimension of A and X.

In order to do this, we restrict somewhat the class of admissible party statutes. In particular, we would like to avoid policy rules that may depend on the choices made by null coalitions of agents. We will, further, assume that parties would react to "small" (but positive in measure) changes in membership with "small" policy changes. This will be guaranteed by the following two–part assumption.

Assumption P1.

(i) (irrelevance of null coalitions) For any $C, C' \in \Sigma$ which differ by a null coalition of agents, we have that $P(C) = P(C')$.

(ii) (continuity) P is continuous when restricted to $\hat{\Sigma}$.

The above assumption allows us to restrict our attention to partitions in $\hat{\Sigma}$ and to insure that the policy proposal profiles induced by partitions in $\hat{\Sigma}$ change continuously with agents' realignment.

At least for membership–based parties, it is not unnatural to assume that, if party membership are on the opposite sides of a hyperplane, the preferred party policies will not be the same.

Assumption P2. (distinct choices) If $C \in \hat{\Sigma}$, then $P^1(C) \neq P^2(C)$.

In other words, P2 says that $P(\hat{\Sigma}) \subset \hat{X}^2$. Finally, we assume that party statutes do reflect preferences of their members. In particular, we would like to avoid parties making policy proposals relatively unpopular among their own members.

Definition 3 *Given a non–null coalition $B \subset A$ and proposals $x, y \in X$ we shall say that x* ***defeats*** *y by a δ–majority, in a (sincere)* ***binary voting*** *(by members of B) if*

$$\frac{F(D_x)}{F(B)} \geq \delta,$$

where $D_x = \{\alpha \in B : u(x;\alpha) \geq u(y;\alpha)\}$.

Assumption P3 (minimal primary support) There exists $\eta > 0$ such that, for any $C \in \hat{\Sigma}$ for which both parties are non–null and for every $i = 1, 2$, the proposal $P^i(C)$ cannot be defeated by a $(1-\eta)$–majority by any other proposal $x \in X$, in a binary voting by members of C_i.[11]

Assumption P3 has a number of significant implications about the policies that can be generated by party statutes. Let K be the convex hull of the

[11] Of course, given the dimension of the policy space n, the minimal support level η can always be chosen sufficiently small so that $(1-\eta)$–majority winners actually exist.

support of F. Since the support of F is compact, so is K. For any $Y \subset R^n$, we let $\operatorname{int} Y$ denote the interior of Y and ∂Y its topological boundary. We have the following result.

Lemma 4 *Suppose assumptions A1, A2 and P3 hold. Then,*

(i) The policy proposal $P^i(C) \in \operatorname{int} K$, for every party $i \in M$ and every partition $C \in \hat{\Sigma}$, such that C_i is non–null.

(ii) The overall policy outcome $T(P(C), w(C)) \in \operatorname{int} K$, for every partition $C \in \hat{\Sigma}$, such that some C_i is non–null.

(iii) There exists a compact subset $K_\eta \subset \operatorname{int} K$, such that for every $C = (C_1, C_2) \in \hat{\Sigma}$ and $i = 1, 2$, we have that $P^i(C) \in K_\eta$, whenever $F(C_i) \geq \frac{1}{2}$.

Proof: Part (i) follows from P3. Assume that, for some $i = 1, 2$, there is C with $F(C_i) > 0$ and such that $P^i(C)$ is on the boundary of K. Choose q so that $(q - P^i(C)) \cdot x \geq 0$ for all $x \in K$ (this is possible, since K is convex).[12] We may take q arbitrarily close to $P^i(C)$. But, this would make the proportion of those party members who prefer $P^i(C)$ to q, relative to the party membership size, arbitrarily small.[13]

Part (ii) follows immediately from (i), since K is convex.

To see Part (iii), suppose $F(C_i) \geq \frac{1}{2}$ for some $i = 1, 2$. By assumption P3, $P^i(C)$ cannot be defeated by any other policy proposal in an $\eta/2$–majority binary voting by all agents in C_i. it follows that $P^i(C)$ cannot be defeated by an $\eta/2$–majority by any other policy proposal in a binary voting by all agents in A. Thus, if $P^i(C)$ were arbitrarily close to the boundary of K, then (as in (i)) we may propose an alternative q arbitrarily close to it.[14] In this way, an arbitrary proportion of agents will prefer q over $P^i(C)$, which contradicts assumption P3. □

We provide next an easy characterization of pluralistic equilibria. Consider the function $\phi = (\phi_1, \phi_2) = (\upsilon_1 \circ P, \upsilon_2 \circ P) = V \circ P$. By assumptions P1 and P2, the statute profile P maps continuously $\hat{\Sigma}$ into $\hat{X}^2$. And, by Lemma 3, V is continuous. Hence, $\phi : \hat{\Sigma} \to \hat{\Sigma}$ is a continuous function, as well.

[12] In other words, q is on the line orthogonal to a supporting hyperplane of the convex hull K at $P^i(C)$.

[13] We are utilizing here the fact that F is finite and hyperdiffuse over compact support.

[14] Take q on the normal to a supporting hyperplane of K at a boundary point which is "close" to $P^i(C)$.

Lemma 5 *If $C^* \in \phi(C^*)$, then $(P(C^*), C^*)$ is a pluralistic two–party equilibrium. Furthermore, $C' \in \Sigma$ may be an equilibrium partition only if it differs by, at most, a null coalition from some $C^* \in \phi(C^*)$.*

Proof: The "if" part is trivial from the definition, since $C^* \in \phi(C^*)$ means that C^* can be supported as a SNE of the voting game between the policy proposals generated by C^*.

Conversely, consider a pluralistic two–part equilibrium partition C'. By Lemma 2, for $C^{'}$ to be a SNE of the voting game it may, at most, be different by the vote of a null coalition from some $C^* \in \hat{\Sigma}$. Because of the irrelevance of null coalitions (assumption (P1(i)) we know that $P(C') = P(C^*)$. Therefore, policy proposals induced by C' would support in SNE only voters' partitions different by no more than a null coalition from those in $\phi(C^*)$. Hence, the "only if" part of the lemma follows. □

We can now state the main result of the paper.

Theorem 6 *If assumptions A1–A2, O1–O3, P1–P3 are satisfied, $m = 2$ and n is odd, then there exists a pluralistic two–party equilibrium.*

The basic idea of the proof is that we may represent pluralistic equilibria as fixed points of a deformation mapping on a compact subset of the cylinder $S^{n-1} \times R$, which is homeomorphic to $S^{n-1} \times [0, 1]$. That the set of such fixed points must be non–empty can be shown by applying Lefschetz's fixed point theorem. (For a survey of the mathematical results involved see McLennan [14] and Munkres [15]). Similar ideas have already been used in Caplin and Nalebuff [9] and Gomberg [12].

The proof is broken up into five parts. Parts I and II characterize the relevant subset $\tilde{\Sigma}$ of $\hat{\Sigma}$; part III defines an auxiliary mapping $\hat{\phi} : \tilde{\Sigma} \to \tilde{\Sigma}$ which coincides with ϕ in the interior of $\tilde{\Sigma}$; part IV shows that $\hat{\phi}$ must have fixed points; and finally, part V shows that all fixed points of $\hat{\phi}$ are in the interior of the domain and, thus, are also fixed points of ϕ. By Lemma 5, these correspond to the required equilibria.

Proof: To avoid possible confusion in the proof, it will be convenient to distinguish between population partitions and the associated hyperplanes. Thus, let us denote by $\Phi(C) = (\Phi_1(C), \Phi_2(C))$ the subsets of the population partition corresponding to the hyperplane $\phi(C)$. We shall also consider the mappings $\pi : \hat{\Sigma} \to S^{n-1}$ and $b : \hat{\Sigma} \to R$ defined by $\pi(C) = \pi^C$ and $b(C) = b^C$.

Part I. We shall prove that there exists $\nu > 0$, such that for every equilibrium partition C and every party $i = 1, 2$ we have that $F(C_i) \geq \nu$. In fact, we will show that there is some $\nu > 0$ such that, if $F(C_i) < \nu$ for some party $i = 1, 2$, then $\Phi_i(C) \cap C_j \neq \emptyset$, for the other party $j \neq i$.

Indeed, suppose otherwise. Then, we can construct a sequence $\left\{C^k\right\}_k = \left\{(C_1^k, C_2^k)\right\}_k$ of population partitions in $\hat{\Sigma}$, such that $F\left(C_1^k\right) \to 0$ and for every integer k, we have that $\Phi_1(C^k) \subseteq C_1^k$.

By Lemma 4, for every $C \in \hat{\Sigma}$, we have that $T(P(C), w(C)) \in K$, a compact set. Hence, we may select a subsequence $\left\{C^k\right\}_k$ (for simplicity, we use the same notation), so that $T(P(C^k), w(C^k))$ converges to a point, say $q \in K$.

By Proposition 1, we see that for each $k \in N$, the point $T(P(C^k), w(C^k))$ is in the hyperplane separating $\Phi_1(C^k)$ and $\Phi_2(C^k)$. And since $\Phi_1(C^k) \subseteq C_1^k$, we must have that $T(P(C^k), w(C^k)) \in C_1^k$, for every $k \in N$. But, $F\left(C_1^k\right) \to 0$ implies that any $x \in \operatorname{int} K$ belongs to C_2^k, for some large enough k. So, we conclude that $q \in \partial K$.

On the other hand, by assumption O3, $g^1(w(C^k)) \to 0$ and $g^2(w(C^k)) \to 1$. Hence, $||T(P(C^k), w(C^k)) - P^2(C^k)|| \to 0$. By part (iii) of Lemma 4, we must have that $P^2(C^k) \in K_\eta$, so $q \in K_\eta$ is bounded away from ∂K, a contradiction.

Part II. Defining the domain. Take

$$\widetilde{\Sigma} = \{C \in \hat{\Sigma} : \frac{\nu}{2} \leq F(C_i) \leq 1 - \frac{\nu}{2}, \quad \text{for } i = 1, 2\}.$$

Since, the support of F is compact, for each $\pi \in S^{n-1}$ we may define

$$\underline{b}(\pi) = \inf\{b^C : \pi^C = \pi, C \in \widetilde{\Sigma}\}$$

and

$$\overline{b}(\pi) = \sup\{b^C : \pi^C = \pi, C \in \widetilde{\Sigma}\}.$$

Thus, for every $C \in \widetilde{\Sigma}$ with the first coordinate $\pi^C \in S^{n-1}$, the intercept b^C belongs to the closed interval $\left[\underline{b}(\pi^C), \overline{b}\left(\pi^C\right)\right]$.

It follows from hyperdiffuseness of F that, both $\underline{b}(\pi)$ and $\overline{b}(\pi)$, are continuous functions of $\pi \in S^{n-1}$. This implies that $\widetilde{\Sigma}$ is homeomorphic to $S^{n-1} \times [0, 1]$, a compact cylinder. Since, $[0, 1]$ is contractible, S^{n-1} is deformation retract of $\widetilde{\Sigma}$.

Part III. The function ϕ may map some partitions from $\widetilde{\Sigma}$ into $\Sigma\backslash\widetilde{\Sigma}$. We shall, therefore, construct a continuous function $\hat{\phi} : \widetilde{\Sigma} \to \widetilde{\Sigma}$ as follows. The mapping $\hat{\phi} = \left(\hat{\phi}_1, \hat{\phi}_2\right)$ is defined by taking $\hat{\phi} \equiv \phi$, when $\phi(C) \in \widetilde{\Sigma}$; but, by replacing each $\phi(C)$ with

$$\hat{\phi}(C) = \begin{cases} (\phi_1(C), \underline{b}(\phi_1(C))) & \text{if } \phi_2(C) < \underline{b}(\phi_1(C)) \\ \left(\phi_1(C), \overline{b}(\phi_1(C)\right) & \text{if } \phi_2(C) > \overline{b}(\phi_1(C)) \end{cases}$$

That is, we let $\hat{\phi}_1 = \phi_1$ and $\hat{\phi}_2 = \max\{\underline{b} \circ \phi_1, \min\{\phi_2, \overline{b} \circ \phi_1\}\}$. Thus, $\hat{\phi}$ inherits the continuity from ϕ.

Part IV. Now we prove that the set $\mathcal{F}\left(\hat{\phi}\right)$ of fixed points of $\hat{\phi}$ is non–empty.

By Lemma 4, we see that $P^i(C) \in \operatorname{int} K \cap C_i$ for every $i = 1, 2$. Hence, the inner product $\pi^C \cdot (P^1(C) - P^2(C)) > 0$. Therefore,

$$\phi_1(C) = \frac{P^1(C) - P^2(C)}{\|P^1(C) - P^2(C)\|} \neq -\pi^C.$$

And we see that $\hat{\phi}_1(C) = \phi_1(C) \neq -\pi^C$, for every $C \in \widetilde{\Sigma}$.

We can chose ν sufficiently small and a point ξ such that all partitions by hyperplanes through it belong to $\widetilde{\Sigma}$. Let Ψ denote this set. Clearly, Ψ is homeomorphic to S^{n-1} and may be parametrized by π^C. Furthermore, one may easily construct a retraction r from $\widetilde{\Sigma}$ onto $\Psi = S^{n-1}$. We can do this, for instance, by identifying Ψ with $S^{n-1} \times \{0\} \subset \widetilde{\Sigma}$ and $\widetilde{\Sigma}$ with $S^{n-1} \times [0, 1]$. Under this identification, the retraction map is $r(\pi, b) = (\pi, 0)$. Consequently, we regard ϕ_1 as a mapping $\phi_1 : \widetilde{\Sigma} \to \Psi$.

Consider now the map $i \circ \phi_1 \circ r : \widetilde{\Sigma} \to \widetilde{\Sigma}$, where i is the inclusion map from $\Psi = S^{n-1} \times \{0\}$ into $\widetilde{\Sigma} = S^{n-1} \times [0, 1]$. Using the above coordinates, we see that $i(\phi_1(r(\pi, b))) = (\phi_1(\pi, 0), 0)$. Since, $\phi_1(C) \neq -\pi^C$ for every $C \in \Psi$, we have that $i \circ \phi_1 \circ r$ is homotopic to the identity map. For instance, a homotopy map $H : \widetilde{\Sigma} \times [0, 1] \to \widetilde{\Sigma}$, between $i \circ \phi_1 \circ r$ and the identity, is given by

$$H((\pi, b), t) = \left(\frac{t\pi + (1-t)\phi_1(\pi, 0)}{||t\pi + (1-t)\phi_1(\pi, 0)||}, tb\right)$$

Therefore, the Lefschetz number, $\Lambda(i \circ \phi_1 \circ r) = \Lambda(\text{identity})$ and the latter coincides with $\chi(\widetilde{\Sigma})$, the Euler characteristic of the space $\widetilde{\Sigma}$ (Munkres [15]). But, since $\widetilde{\Sigma}$ is homeomorphic to the sphere $S^{n-1} \times [0, 1]$, its Euler characteristic is $\chi(\widetilde{\Sigma}) = \chi(S^{n-1}) = 2$, for odd n.

In addition, there is a homotopy $\widetilde{H} : \tilde{\Sigma} \times [0,1] \to \tilde{\Sigma}$, between the mappings $\hat{\phi}$ and $i \circ \phi_1 \circ r$, given by

$$\widetilde{H}\left(\left(\pi, b\right), t\right) = \left(\phi_1\left(\pi, bt\right), t\hat{\phi}_2\left(\pi, bt\right)\right).$$

Hence, $\Lambda\left(\hat{\phi}\right) = \Lambda\left(i \circ \phi_1 \circ r\right) = 2$, does not vanish. By the Lefschetz fixed point theorem (McLennan [14]), $\mathcal{F}\left(\hat{\phi}\right) \neq \emptyset$.

Part V. We show, finally, that $\mathcal{F}\left(\hat{\phi}\right) \subset \text{int}\left(\Sigma\right)$.

Suppose otherwise. Then, we can find a partition $C^* \in \partial\widetilde{\Sigma}$ such that $\hat{\phi}\left(C^*\right) = C^*$. Hence, $F(C_i^*) = \nu/2$ for some $i = 1, 2$. Say, $F(C_1^*) = \nu/2$. But then, we have that $F(C_1^*) < \nu$ and $\Phi_1\left(C\right) \cap C_2^* = C_1^* \cap C_2^* = \emptyset$, which contradicts the claim in part I.

To finish the proof of the Theorem, note that $\hat{\phi}(C) = \phi(C)$ as long as $C \notin \partial\widetilde{\Sigma}$. Thus, Part V implies that $\mathcal{F}\left(\hat{\phi}\right) = \mathcal{F}\left(\phi\right)$, and we conclude that $\mathcal{F}\left(\phi\right) \neq \emptyset$. □

The existence result obtained above for the odd dimensional policy spaces is quite general, in that it imposes relatively few restrictions on the internal policy rules of the parties.

Unfortunately, when the dimension of the policy space is even, we may only achieve more limited results. The following example shows an even dimensional model satisfying all our assumptions and for which there is no Pluralistic Equilibrium. The construction of this example is closely related to the well known fact that, there exists a continuous non vanishing vector field on the sphere S^k, if and only if k is odd.

Example 1 *There are two political parties, $M = \{1, 2\}$. For simplicity, we assume that the policy space $X = \{x \in R^{2n} : ||x - z|| \leq 1\}$ is a closed disc of dimension $2n$ with center $z \in R^{2n}$. The boundary of X is S^{2n-1} an odd dimensional sphere. For this example, one only needs that X is a compact, convex set whose boundary is homeomorphic to the sphere S^{2n-1}.*

Let Y be a continuous non–vanishing vector field on S^{2n-1}. We define the policy rule as follows. Let $C \in \hat{\Sigma}$ be a population partition and let $H(C)$ be the hyperplane that separates C_1 and C_2. The intersection of $H(C)$ with X is a $2n - 1$ dimensional closed disk with center $Z\left(C\right)$. Let $r(C)$ be the line through $Z(C)$ orthogonal to $H(C)$. The line $r(C)$ intersects the boundary of X in two opposite points $e_i \in C_i$, $i = 1, 2$. Let $d^i(C) \in C_i$, for $i = 1, 2$ be the *midpoint on $r(C)$ between $Z(C)$ and e_i. For each $C \in \hat{\Sigma}$, the points $d^i(C)$ are in the interior of C_i.*

We construct now a continuous map $g : \hat{\Sigma} \to R_{++}$ such that, for every partition $C = (C_1, C_2)$ and for each coalition $i = 1, 2$, the point $d^i(C) \pm g(C)Y(e_i)$ is in the interior of C_i. We let now $p^1(C) = d^1(C) + g(C)Y(e_1)$ and $p^2(C) = d^2(C) - g(C)Y(e_1)$. One checks easily that the functions $p^1(C)$ and $p^2(C)$ are continuous.

By construction, for any given partition C, the segment $[p^1(C), p^2(C)]$ is not perpendicular to the hyperplane $H(C)$. Thus, no pair of the form $(P(C), C)$ can be a pluralistic multi–party equilibrium.

5 The Mean voter rule

While the example in the previous section highlights the difficulty of establishing general existence results when there is an even number of policy dimensions, we may still be able to obtain more limited results for some interesting classes of policy rules. As an important example we shall show that, when policies are two–dimensional and each party chooses for its proposal the ideal point of its mean member, a two–party equilibrium must exist.

Let us assume that the individual distribution on A is given by the non–atomic measure F with a density $f(x)$ and, for simplicity, identify the set of agents with the set of policies, i.e., $X = A \subset R^2$. As above, there are two parties and the weights are given by the total population in each coalition

$$\omega(C) = (\omega^1(C_1), \omega^2(C_2)) = \left(\int_{C_1} f(x)\, dx, \int_{C_2} f(x)\, dx \right)$$

But now, we specify that the party policy choice of each coalition $i = 1, 2$ is its center of gravity

$$P^i(C_i) = \frac{1}{\omega^i(C_i)} \int_{C_i} x f(x)\, dx = \frac{1}{\omega^i(C_i)} \left(\int_{C_i} x_1 f(x)\, dx, \int_{C_i} x_2 f(x)\, dx \right) \quad (1)$$

where $x = (x_1, x_2) \in X$ and we have assumed that $\omega^i(C_i) \neq 0$. To avoid cumbersome notation, let us write $p_i = P^i(C_i)$, $\omega^i = \omega^i(C_i)$, for $i = 1, 2$, when there is no possibility of confusion.

The outcome function $T(p, \omega) = \omega^1(C_1)p^1 + \omega^2(C_2)p^2 = \int_X x f(x)\, dx$ therefore results in a policy outcome which is independent of the partition. After a translation, we may (and will) assume that $T(p, \omega) = (0, 0) \in X$.

With this convention, p^1 and p^2 are collinear, pointing in opposite directions. In general, $p^1 - T(p,\omega)$ and $p^2 - T(p,\omega)$ are collinear, pointing in opposite directions.

Proposition 7 *Suppose that $n = 2$ and the policies chosen by the parties are given by the rule in Equation 1. Then, there is a partition $C = (C_1, C_2)$ such that $(P(C), C)$ is a pluralistic equilibrium.*

Proof: By Proposition 1, the Strong Nash Equilibria correspond to coalitions C_1 and $C_2 = X \setminus C_1$ which are separated by a straight line $H(C)$ containing the point $T(p,\omega) = (0,0)$. We identify the unit vector, say q, orthogonal to the line $H(C)$ with the partition $C = \{C_1, C_2\}$ so,

$$C_1 = \{x \in X : q \cdot x \leq 0\},\ C_2 = \{x \in X : q \cdot x > 0\}.$$

Since X is convex, by changing to polar coordinates we may write

$$X = \{(r\cos\theta, r\sin\theta) : 0 \leq r \leq r(\theta), 0 \leq \theta < 2\pi\}$$

for some function $r : [0,\ 2\pi) \to R^n_+$. And we see that, the partition induced by the vector $q = (\cos\alpha, \sin\alpha)$ is then,

$$C_1 = \{(r\cos\theta, r\sin\theta) : 0 \leq r \leq r(\theta), \alpha + \frac{\pi}{2} \leq \theta < \alpha + \frac{3\pi}{2}\}$$

$$C_2 = \{(r\cos\theta, r\sin\theta) : 0 < r \leq r(\theta), \alpha - \frac{\pi}{2} < \theta < \alpha + \frac{\pi}{2}\}$$

We remark that $q = (\cos\alpha, \sin\alpha) \equiv C = \{C_1, C_2\}$ is a pluralistic explicitly equilibrium if and only if q is collinear with the vectors p^1 and p^2, that is if and only if p^1 and p^2 are orthogonal to the half–line

$$\{r(\cos\theta, \sin\theta) : 0 \leq r < \infty, \theta = \alpha + \frac{\pi}{2}\} = \{r(-\sin\alpha, \cos\alpha) : 0 \leq r < \infty\}.$$

Thus, the problem is reduced to show that we can find $\alpha \in [0,\ 2\pi)$ such that $G(\alpha) = 0$, where

$$G(\alpha) = -\sin\alpha \int_{C_1} x_1 f(x)\,dx + \cos\alpha \int_{C_1} x_2 f(x)\,dx$$

Changing to polar coordinates, and using Fubini's Theorem, this is the same as

$$-\int_{\alpha+\frac{\pi}{2}}^{\alpha+\frac{3\pi}{2}} \int_0^{r(\theta)} r^2 \sin\alpha \cos\theta f(r,\theta)\, dr\, d\theta + \int_{\alpha+\frac{\pi}{2}}^{\alpha+\frac{3\pi}{2}} \int_0^{r(\theta)} r^2 \cos\alpha \sin\theta f(r,\theta)\, dr\, d\theta =$$

$$\int_{\alpha+\frac{\pi}{2}}^{\alpha+\frac{3\pi}{2}} \int_0^{r(\theta)} \sin(\theta-\alpha) r^2 f(r,\theta)\, dr\, d\theta = \int_{\alpha+\frac{\pi}{2}}^{\alpha+\frac{3\pi}{2}} \sin(\theta-\alpha) g(\theta)\, d\theta$$

with

$$g(\theta) = \int_0^{r(\theta)} r^2 f(r,\theta)\, dr.$$

Note that $g(\theta+2\pi) = g(\theta)$. Making the change of variable $\theta = t+\alpha$, we see that $G(\alpha)$ equals

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} g(t+\alpha) \sin t\, dt,$$

Integrating now the function $G(\alpha)$ and using again Fubini's Theorem, we obtain

$$\int_0^{2\pi} G(\alpha)\, d\alpha = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin t \left(\int_0^{2\pi} g(t+\alpha)\, d\alpha\right) dt$$

But

$$\int_0^{2\pi} g(t+\alpha)\, d\alpha = \int_0^{2\pi} g(\alpha)\, d\alpha = A$$

is independent of t, because $g(\theta+2\pi) = g(\theta)$. Hence,

$$\int_0^{2\pi} G(\alpha)\, d\alpha = A \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin t\, dt = 0$$

and, by continuity of G, there must be some $\alpha \in [0,\, 2\pi)$ such that $G(\alpha) = 0$.

This result is, admittedly, more limited than the one obtained in the previous section for the odd–dimensional case. Nonetheless, in the present setting the mean–voter rule has some interest. Since, Caplin and Nalebuff [9] have shown that, under some conditions on the distribution of individuals, the mean–voter proposal cannot be defeated by any other alternative by a qualified majority in a binary vote. Also, Grofman and Feld [10] have proved that, for $n = 2$, the winner under the mean voter rule coincides with the winning proposal under the (generalized) Borda count.

6 Robustness of equilibria.

Example 1 above illustrates the potential non–robustness of some equilibria to small changes in the model: an arbitrarily small change in party statutes may, potentially be sufficient to completely destroy some equilibria. Since we are unlikely to be able to observe precisely political decision processes, both within parties and in the society as a whole, a model which depends in a discontinuous way on changes in the specifications of these processes is not likely to result in reliable predictions. Fortunately, for the most part it is possible to guarantee the robustness of equilibria implied by Theorem 6.

The robustness concept employed in this paper follows that in Gomberg [12].

Definition 4 *A compact set of equilibrium partitions $B \subset \hat{\Sigma}$ of a two–party model, with a profile P of party statutes, is* ***robust to small party statute changes***[15] *if, for any open neighborhood $U \subset \hat{\Sigma}$ of B, there exists an open neighborhood $W \subset C(\hat{\Sigma}, X)$ of P, such that for any perturbed model with party statutes' profile $P' \in W$, there exists an equilibrium of some population partition $S' \in U$. A robust set of equilibrium partitions is called* ***minimal*** *if it has no robust proper subset.*

It can be shown that, when n is odd, the assumptions of Theorem 6 guarantee robustness of the party statutes.

Theorem 8 *Under the same assumptions as in Theorem 6 (so, in particular, n is odd), there exists a non–empty compact connected minimal robust set of equilibrium partitions.*

The proof of this result is identical to the proof of the robustness result in Gomberg [12]. Indeed, it can be shown that if, for any $P \in C(\hat{\Sigma}, X)$, we denote $\phi_P = V \circ P$, then for any open neighborhood W of ϕ_P we can find an open neighborhood U of P such that for any $P' \in U$ we have $\phi_{P'} \in W$. But, since all the assumptions we needed for the existence result of Theorem 6 are satisfied, this implies that ϕ_P has a minimal essential set of fixed points (see McLennan [14]). It follows that there exists a minimal robust set of equilibria in this model.[16]

[15] A similar refinement could be defined with respect to small changes in the final outcome rule T.

[16] For details see Gomberg [12].

Unfortunately, the equilibria provided by the mean voting rule in Proposition 7 are not always robust. Indeed, if the population distribution F is uniform, it can be easily seen that **any** population partition by a hyperplane through $\int_X xf(x)\,dx$ corresponds to an equilibrium. However, if, for instance, adherence to the mean–voter rule within each party is not perfect, but rather incorporates an arbitrarily small systematic bias, we may destroy **all** of these. As an example, suppose the chosen policy in each party is subject to an arbitrarily small "counter–clockwise" tremble

$$\begin{aligned} p^i &= \frac{1}{\omega^i(C_i)} \int_{C_i} xf(x)\,dx \begin{pmatrix} \sin\left(\frac{\pi}{2}-\varepsilon\right) & \cos\left(\frac{\pi}{2}-\varepsilon\right) \\ -\cos\left(\frac{\pi}{2}-\varepsilon\right) & \sin\left(\frac{\pi}{2}-\varepsilon\right) \end{pmatrix} = \\ &= \frac{1}{\omega^i(C_i)} \left(\int_{C_i} x_1\,dx, \int_{C_i} x_2 dx \right) \begin{pmatrix} \sin\left(\frac{\pi}{2}-\varepsilon\right) & \cos\left(\frac{\pi}{2}-\varepsilon\right) \\ -\cos\left(\frac{\pi}{2}-\varepsilon\right) & \sin\left(\frac{\pi}{2}-\varepsilon\right) \end{pmatrix}, \; i = 1,2 \end{aligned}$$

for an arbitrarily small $\varepsilon > 0$. One can observe, that in this case no population partition may be an equilibrium, since for any partition the induced policies will cause a slight "rotation" of the SNE partition hyperplane.

Fortunately, however, it turns out that this is, essentially, the only possible example of non–robustness.

Proposition 9 *If $n = 2$, the party platforms are determined by a mean voter rule, and there exists a disequilibrium partition C by a hyperplane through $\int_X xf(x)\,dx$, then there exists a non–empty compact connected minimal robust set of equilibrium partitions B^*.*

Proof. Consider the associated function $\phi : \hat{\Sigma} \to \hat{\Sigma}$. By proving the existence of equilibrium, we have shown that it has at least one fixed point. We shall now show that there is at least one set of essential fixed points. As in Gomberg [12], this will imply that there exists at least one minimal robust set of equilibria.

We have shown that ϕ maps every partition in $\hat{\Sigma}$ into a partition by a hyperplane through $\int_X xf(x)\,dx$. The set of such partitions, which we shall denote as Σ^*, is clearly homeomorphic to S^1. Consider the restriction $\phi|_{\Sigma^*}$ of ϕ to Σ^*. Since the intercept is fixed, we shall only be concerned with the slope of the partition hyperplane, which we shall parametrize by $\alpha \in [0, 2\pi)$. Thus, subject to the choice of coordinates, we may write

$$\phi|_{\Sigma^*}(\alpha) = \frac{P^1(\alpha) - P^2(\alpha)}{||P^1(\alpha) - P^2(\alpha)||}$$

and a partition is an equilibrium if and only if it differs by at most a null coalition from a partition such that

$$\alpha = \phi|_{\Sigma^*}(\alpha).$$

In fact, in the proof of Proposition 7, we have effectively shown that the fixed points of $\phi|_{\Sigma^*}$ are solutions to the equation

$$G(\alpha) \equiv (-\sin\alpha, \cos\alpha) \cdot P^1(\alpha) = 0,$$

where the continuous function $G : [0, 2\pi] \to R$ satisfies $G(0) = G(2\pi)$ and $\int_0^{2\pi} G(\alpha)\, d\alpha = 0$

Since, not all points on Σ^* correspond to an equilibrium, without loss of generality, we may choose the coordinates so that $G(0) > 0$. Therefore, there must exist a point $\hat{\alpha} \in (0, 2\pi)$ such that $G(\hat{\alpha}) < 0$, which, in turn, implies that there exists a minimal essential set $\emptyset \neq [a, b] \subset (0, \hat{\alpha})$ of zeroes of G, which is "stable" under the fictitious dynamics implied by G. It can be easily seen that this, in turn, implies "stability" of the corresponding set of fixed points $B \subset \Sigma^*$ of $\phi|_{\Sigma^*}$.

Consider now a small neighborhood $U \subset \hat{\Sigma}$ of B. Clearly, B is "stable" under the fictitious dynamics implied by $\phi : \hat{\Sigma} \to \hat{\Sigma}$ and, therefore, $ind(B) = 1 \neq 0$. □

7 Conclusions and Further Research

In this paper we have considered a model of endogenous formation of political parties and have provided sufficient conditions for the existence of equilibrium. It turns out that, in general, we are able to achieve these results for multiple dimensions of the party platform policy space. Furthermore, at least some equilibria are shown to be robust to small errors in the specification of the model.

We believe that there are two main reasons to regard this work as relevant. Firstly, there has been recently a large literature on Political Economy assuming ideological political parties. In most cases, however, such party ideology is given exogenously, and in the few exceptions where that is not the case (as in [6], [17], [20] and [22]), the assumptions are too restrictive. Even though, the model in this paper also requires some strong assumptions, it provides a more general theory, which goes well beyond the level of "specific examples" analyzed in those related works. Moreover, our assumptions accommodate without any difficulties the setting in which parties are unable to commit on their proposals. Many recent papers on Political Economy (see [3], [4] and [19]) justify and use this as a valid assumption in modeling the political competition.

Secondly, the paper provides new insights on the relationship between the dimensionality of the policy space and the existence of equilibrium. This relationship is often seen as a negative one: the higher the dimension of the policy space is, the harder it is to guarantee existence of equilibrium. This view is a consequence of the well known results in the classical models of political competition, where existence of equilibrium is very rare in two and higher dimensions (an exception is the model in [21]). We have shown, however, that when ideology and membership of the parties are endogenous to the model, it is harder to obtain existence in the two–dimensional case than in three dimensional one (our results are general for the even and odd dimensions. It is, however, very unusual to assume policy spaces of dimension four or higher). Moreover, this apparently paradoxical result, that recalls the one established by Caplin and Nalebuff [9], is not due to any artificial mathematical artifact used in the model. This, however, does not imply that existence of equilibrium in the two-dimensional policy space is always impossible and we indeed show existence for the particular, but important, case in which the ideology of the party coincides with the mean ideology of its members.

There remain a number of important theoretical directions to be explored in this class of models. One natural next step is extending the current results for the case of more than two political parties. Another promising direction for future research is expanding the definition of the party statutes to model accommodate the strategic behavior of parties.

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