

Dynamics of Networks If Everyone Strives for Structural Holes

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DRAFT:

November 15, 2004

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Abstract

Ronald Burt's 'constraint' measure was introduced (1992) to capture the information and control benefits of actors in social networks. A series of empirical studies have demonstrated how this measure positively relates to a wide range of indicators of social success. In this study it is assumed that all actors strive for structural holes by adding links that have net control and information benefits and by deleting costly links. We prove various theorems on stability and efficiency criteria of these/resulting networks from a recently emerged literature on dynamic networks. In addition, we run simulations to investigate the relative frequencies with which certain equilibrium networks are obtained from a random initial network. An important class of stable and efficient networks turns out to be balanced complete bipartite networks.

Introduction

There are a few well-established ideas in sociology about how positions in social networks are of use to the people occupying them. One idea is that those positions whose neighbors do not know one another tend to have access to a richer supply of information and have more control over their network relations. Since Ronald Burt's formalization of this idea of "structural holes" (Burt 1992), a number of studies have appeared that empirically establish the hypothesized relationship between quality of network position and quality of life.² Producer profit margins are larger for firms spanning structural holes in buyer-supplier networks (Burt et al., 2002; Talmud 1994; Yasuda 1996; Jang 1997). Jobs are found faster through ties that connect otherwise disconnected groups (Granovetter 1995 [1974]) and the jobs found are more desirable (Lin 1999; Bian 1994; Leenders & Gabbay 1999; Lin, Cook, & Burt 2001). Salaries are higher for managers with higher structural hole scores (Burt 1995, 1997, 1998, 2000; Podolny and Baron 1997; Burt, Hogarth, and Michaud 2000; Mehra, Kilduff, and Brass 2000; Mizruchi and Sterns 2001). And structural holes are positively correlated with income, positive performance evaluations, peer reputations, promotions, and good ideas (Gabbay 1997; Burt 2001, 2004).

Whether in general scientific study leaves the behavior that it studies untouched, is open to debate and dependent on method and context. But on the website <http://www.onlinebusinessnetworks.com/blog/2004/06/11/on-structural-holes> under "Online Business Networking Tips" we find David Teten, CEO of an independent research firm, recommending managers to follow Burt's advice by "...building strong ties with just Hanz or just Franz, and not spending too much time building ties with the other person. We believe the control and information benefits you can get from Hanz and Franz are redundant...". Similar business tips can be found on the following first couple of Google hits on a 'structural holes' query:

http://www.venchar.com/2004/05/structural_hole.html

http://ming.tv/flemming2.php/_show_article/_a000010-001292.htm

<http://www.brint.com/wwwboard/messages/12412.html>

http://www.meshforum.org/archives/links_network_discussion/structural_holes_and_gaps_in_your_network.html

Thus, Ronald Burt has been able to spread his idea not only among scientists but also among practitioners. A few managers like Teten have adopted the idea and have started to optimize their network position. And Burt's theory and the before-mentioned empirical evidence suggest that they must have gained from these efforts. But it does not stop there. Teten tells other managers that they should do the same. And information that comes from him must spread extra fast. Anticipating these consequences, we ask what would happen if *everyone* strived for structural holes. In this paper, we consider groups of people who have realized the benefits from occupying information-rich positions and have come to strategically act upon this knowledge. They add links that enhance their structural advantage, and sever those that hamper it. We are interested in what networks evolve under such a scenario and whether they are socially efficient.

² See Burt (2002) for a more extensive review.

Recently, social scientists have started to develop models of network dynamics in which actors strategically manipulate links (Leik 1991, 1992; Doreian & Stokman 1997; Bala & Goyal 2000a; Watts 2001; Bonacich 2002a, 2002b; Dutta & Jackson 2003; Snijders 2004; Van de Rijt & Willer 2004). These models take previously developed theory on which networks benefit which actors as given, and produce predictions about which network actors would end up in if they added and severed links in order to maximize their network benefits. Two kinds of models are often distinguished: Models of one-sided link formation and models of two-sided link formation. In models of the former type, ties are directed and actors can unilaterally add and delete ties *to* others. In models of the latter type, ties are undirected and actors can unilaterally delete ties *with* others, but for the addition of a tie she needs permission of the respective other actor. For a model of either type, it is specified how actors value network positions. This specification is referred to as ‘value function’ or ‘utility function’. Many value functions that have been explored are based on some structural advantage in communication or information flow (Bala & Goyal 2000a, 2000b; Dutta & Jackson 2000; Jackson & Wolinsky 1996; Johnson & Gilles 2003; Slikker & Van den Nouweland 2003). In other cases, utility represents bargaining potential (Leik 1991, 1992; Kranton & Minehart 2000, 2001; Corominas-Bosch 1999; Willer & Willer 2000; Bonacich 2002a, 2002b; Bloch & Ghosal 2003). In yet other cases, networks represent time invested in cooperative projects, utility measuring the returns on these investments (Gehrig, Regibeau, & Rockett 2003; Jackson & Wolinsky 1996).

In this paper, we explore what happens if actors are concerned with structural holes. All actors attempt to minimize Burt’s ‘network constraint’ measure (Burt 1992: 54-55), which constitutes their value function. In the theoretical examples and empirical applications of the idea of structural holes, ties between actors either represent time spent together or a history of economic transactions. These are activities that require joint consent and can be unilaterally withdrawn from. Our model is therefore of the two-sided link formation type. We address two questions: What kinds of networks emerge? Which networks are more likely to emerge given a random starting network? We derive answers to these questions both analytically and through simulation.

The Model

Let $n \geq 2$ be the number of actors, $N = \{1, 2, \dots, n\}$ the set of actors, and $g^N = \{ij \mid i \neq j; i, j \in N\}$ the complete graph of undirected connections ij between the n actors. A network $g \subseteq g^N$ of n actors is represented by an $n \times n$ symmetric adjacency matrix X , with $x_{ij} = x_{ji} = 1$ if $ij \in g$ and $x_{ij} = x_{ji} = 0$ if $ij \notin g$. We consider non-reflexive networks, so $x_{ii} = 0$ for all i . Let $d_i = \sum_j x_{ij}$ be

actor i ’s degree and define $p_{ij} = \frac{x_{ij}}{d_i}$. The network constraint c_i of a connected actor i ($d_i > 0$)

equals $c_i \equiv \sum_{j \neq i} x_{ij} \left(p_{ij} + \sum_{k \neq i, k \neq j} p_{ik} p_{kj} \right)^2$. At some points in the paper where this is convenient,

we use an equivalent definition:

$$c_i \equiv \frac{1}{d_i^2} \sum_j \left[1 + \sum_k \frac{1}{d_k} \right]^2$$

where j is the index for neighbors of i and k is the index for neighbors of i that are also connected to j . This definition is equivalent, because in the original definition, for non-neighbors of i , the summation element that concerns this actor equals 0. For all neighbors of i , $p_{ij} = d_i^{-1}$.

We set $c_i = 2$ for disconnected actors ($d_i = 0$), which is larger than the highest possible constraint for connected actors.

Theorem 0. The Burt constraint $c_i \leq 9/8$ if $d_i > 0$.

Proof. See appendix.

Thus, by setting $c_i = 2$, we assume that being connected is always better for an actor than being an isolate. Lastly, we consider a network payoff function $u_i(g) = 2 - c_i$ such that $u_i(g)$ is always positive for connected actors and equals 0 for disconnected actors. Effectively we will only use that the utility of a network position decreases with the network constraint to find stable networks.

We first introduce some additional notation. Let $g + ij = g \cup ij$ and $g - ij = g \setminus ij$. Let $N^d(i, g) = \{j \in N \mid ij \in g\} \subset N$ be the set of neighbors of actor i , and $h_M^i = \{ij \in g \mid j \in M\}$ any subset $M \subseteq N \setminus \{i\}$. Following Jackson (2003) and Gilles and Sarangi (2004) we consider pair-wise stability, strong pair-wise stability, and strong stability. We add what we call ‘‘unilateral stability’’, which is stronger than strong pair-wise stability, but weaker than strong stability. We formally define each of the concepts and briefly explain them in natural language.

Definition 1. A network $g \subseteq g^N$ is *pair-wise stable* (Jackson 2003: 21) if

- (i) $\forall ij \in g, u_i(g) \geq u_i(g - ij) \wedge u_j(g) \geq u_j(g - ij)$
- (ii) $\forall ij \notin g, (u_i(g) = u_i(g + ij) \wedge u_j(g) = u_j(g + ij)) \vee u_i(g) > u_i(g + ij) \vee u_j(g) > u_j(g + ij)$

A network is pair-wise stable if (i) for each present tie, its deletion makes both actors involved in it weakly worse off, and (ii) for each absent tie, its addition either leaves both actors involved in that tie equally well off or makes at least one of the two strictly worse off.

Definition 2. A network $g \subseteq g^N$ is *strongly pair-wise stable* (Gilles and Sarangi 2004: 13) if

- (i) $\forall i \in N, \forall M \subseteq N^d(i, g), u_i(g \setminus h_M^i) \leq u_i(g)$
- (ii) $\forall ij \notin g, (u_i(g) = u_i(g + ij) \wedge u_j(g) = u_j(g + ij)) \vee u_i(g) > u_i(g + ij) \vee u_j(g) > u_j(g + ij)$

A network is strongly pair-wise stable if (i) for each subset of all ties of some actor, deleting it would make her weakly worse off, and (ii) for each absent tie, its addition either leaves both actors involved in that tie equally well off or makes at least one of the two strictly worse off.

Definition 3. A network $g' \subseteq g^N$ is *obtainable* from $g \subseteq g^N$ by $S \subseteq N$ (see Jackson and van den Nouweland 2003; Jackson 2003: 12) if

- (i) $\forall ij \in g' \setminus (g' \cap g), \{i, j\} \subseteq S$
- (ii) $\forall ij \in g \setminus (g' \cap g), \{i, j\} \cap S \neq \emptyset$

One network is obtainable from another by a certain coalition if (i) for each tie that is added, both actors involved in the tie are members of the coalition, and (ii) for each tie that is deleted, at least one of the actors involved in the tie is member of the coalition.

Definition 4. A network $g' \subseteq g^N$ is *unilaterally obtainable* from g by i through $S \subseteq N \setminus \{i\}$ if

- (i) $\forall jk \in g' \setminus (g' \cap g), (j = i \vee k = i) \wedge \{j, k\} \subseteq S + i$
- (ii) $\forall jk \in g \setminus (g' \cap g), (j = i \vee k = i)$

One network is unilaterally obtainable from another by a central actor and through a certain coalition, if each tie that is added or deleted involves the central actor and if each tie that is added involves a member of the coalition.

Definition 5. A network $g' \subseteq g^N$ is *unilaterally stable* if $\forall i, \forall S \subseteq N \setminus \{i\}, \forall g' \subseteq g^N$ unilaterally obtainable from g by i through S such that $u_i(g') > u_i(g), \exists j \in S$ such that $ij \in g' \wedge ij \notin g \wedge u_j(g') < u_j(g)$.

Unilateral stability excludes the possibility that some actor proposes a new network that is identical to the old network with respect to all ties that do not involve that actor, thereby making herself better off while making no actor that she connects to worse off.

Definition 6. A network $g \subseteq g^N$ is *strongly stable* if $\forall S \subseteq N, \forall i \in S, \forall g' \subseteq g^N$ obtainable from g through S such that $u_i(g') > u_i(g), \exists j \in S$ such that $u_j(g') < u_j(g)$.

Strong stability excludes the possibility that some coalition adds ties that only involve coalition members and deletes ties that involve at least one coalition member and then ends up making no coalition member worse off and at least one strictly better off.

The four stability concepts are listed in decreasing order of generality. Strong stability implies unilateral stability, because unilaterally obtainable is the special case of obtainable in which all added and deleted ties have a single actor in common. Unilateral stability implies strong pair-wise stability, because all networks obtained through either the addition of a tie that involves a particular actor or the deletion of some subset of all ties that involve that actor are a special case of the set of unilaterally obtainable networks. Strong pair-wise stability implies pair-wise stability, because the deletion of any tie that involves an actor is a special case of the deletion of any subset of all ties that involve that actor.

Unilateral stability resembles *monadic stability* as defined by Gilles and Sarangi (2004: 22). The difference between the two concepts is that in the case of unilateral stability, a coalition member blocks a transition to *the network that the central actor proposes*, while in the case of monadic

stability, she blocks the transition to *a network with only her tie with the central actor added*. In the latter case, the coalition members compare utility in the current network with utility in a network that is not necessarily the same as the one that the central actor is proposing. Indeed, a strongly stable network could be monadically unstable because of a transition to another network being permitted by a coalition of which some members are strictly worse off in the new network (see Gilles and Sarangi (2004:24) for an example of such a network).

In addition to this undesirable property, we believe unilateral stability, not monadic stability, is the closest one can get to an equivalent of *individual stability* (Gilles & Sarangi 2004: 13) for models of two-sided link formation. Individual stability was introduced for models of one-sided link formation to indicate a network that represents a combination of tie addition and deletion strategies that constitutes a Nash Equilibrium. In a unilaterally stable network, no actor can unilaterally improve her utility through tie deletion and tie addition that is permitted by the two-sided link formation rule.

The main aim of this paper is to give some characterizations of the stable and efficient networks if actors only care about their network constraint. Emphasis is placed on pair-wise stability and Pareto-optimality respectively. In addition, we want to find out which networks are more likely to evolve if we start from an arbitrary network and let rational actors play our game. To reach the second aim we will run simulations on networks varying from size 2 to 25.

Analytic Results

In this paragraph we provide some analytic results on stability and efficiency of particular network structures.

Theorem 1. The shortest path between any pair of actors in a pair-wise stable network has length smaller than or equal to 2.

Proof. Recall from the previous section that network constraint of actor i , c_i , can be rewritten as follows:

$$c_i \equiv \frac{1}{d_i^2} \sum_j \left[1 + \sum_k \frac{1}{d_k} \right]^2$$

where j is the index for neighbors of i and k is the index for neighbors of i that are also connected to j . An additional tie between two actors at distance larger than 2 creates no new closed triad. Assume actor i is one of the actors that creates the tie. Let c_i^* denote the network constraint of i after the initiation of the new tie, and let j be the index for neighbors in the old network. We have

$$\begin{aligned} c_i^* - c_i &= \frac{1}{(d_i + 1)^2} \left[1 + \sum_j \left(1 + \sum_k \frac{1}{d_k} \right)^2 \right] - \frac{1}{d_i^2} \sum_j \left(1 + \sum_k \frac{1}{d_k} \right)^2 = \frac{1}{(d_i + 1)^2} + \frac{d_i^2 - (d_i + 1)^2}{d_i^2 (d_i + 1)^2} \sum_j \left(1 + \sum_k \frac{1}{d_k} \right)^2 \\ &\leq \frac{1}{(d_i + 1)^2} + \frac{d_i^2 - (d_i + 1)^2}{d_i (d_i + 1)^2} = -\frac{1}{d_i (d_i + 1)} < 0 \end{aligned}$$

The addition of the new link necessarily decreases actor i 's network constraint, and hence increases her utility. The same reasoning holds for the other actor involved in the new tie.

Corollary 1. Creating a tie without creating closed triads is always beneficial for both actors involved in the new tie.

Corollary 2. A network of disconnected parts cannot be pair-wise stable (since any pair of actors in two separate parts are always willing to connect).

Definition 7. An m -partite network is a network in which the actors can be divided into m groups such that there are no ties within the groups. A *complete m -partite* network is an m -partite network in which all the possible links between the actors in the m groups exist (see Wasserman & Faust 1994: 120). If $m = 2$, m -partite networks are called bipartite networks.

Corollary 3. Pair-wise stable networks that are bipartite networks are necessarily complete bipartite networks (otherwise some actors are at a distance larger than 2).

Definition 8. A *star* is a complete bipartite network in which one of the groups consists of only one actor.

Theorem 2. Suppose we have a complete bipartite network with k actors in one group and $l = n - k$ actors in the other group. Then this complete bipartite network is a pair-wise stable network, unless it is a star with $n > 4$.

Proof. See Appendix.

Definition 9. A network is *Pareto-optimal* if there is no other network such that no actor is worse off and at least one actor is better off.

Note that strong stability implies Pareto-optimality, because any transition away from a Pareto-optimal network would be blocked by at least one in the coalition of all actors.

Theorem 3. If a network is a complete bipartite network, then it is Pareto-optimal.

Proof. See Appendix.

Although we did not find any counter-example in the simulation results in the next section, we were not able to prove the following conjecture.

Conjecture 1. A network is Pareto-optimal if and only if it is a complete bipartite network.

Definition 10. A bipartite network is *balanced* if $l - k \leq 1$.

Theorem 4. A complete bipartite network is unilaterally stable if and only if it is balanced.

Proof. See Appendix.

Corollary 4. A complete bipartite network is strongly stable if and only if it is balanced.

The following conjecture follows directly from conjecture 1. This would be a very nice result, because it would imply the existence of a unique equilibrium under the strongest stability assumption. [Another conjecture that would be sufficient for conjecture 2 is that unilaterally stable networks cannot have triads. Maybe one should think of a constructive proof that shows how an individual in a triad can always improve her position].

Conjecture 2. A network is strongly stable if and only if it is a balanced complete bipartite network.

Theorem 5. All complete m -partite networks with equally sized groups are pair-wise stable if $m \geq 2$ and the size of the group, say $n_2 = n / m$, is also at least 2 (this excludes the full and the empty network).

Proof. See Appendix.

Definition 11. A *complete* network is a network in which everyone has degree $n - 1$.

In other studies on network dynamics, special networks such as the complete network, stars, “wheels,” and “cycles” are often found to be stable and/or efficient (e.g., Bala and Goyal 2000). For the Burt constraint measure, these networks are neither stable nor efficient except for some specific small examples. Stars are already discussed before because they are a special case of bipartite networks. Further results are presented below.

Theorem 6. The complete network is not pair-wise stable if $n > 2$.

Proof. Let c_i be actor i 's network constraint in the full $n > 2$ -actor network and c_i^* her constraint after deleting one link. We have

$$c_i^* - c_i = (n-2) \left[\frac{2n-4}{(n-1)(n-2)} \right]^2 - (n-1) \left[\frac{2n-3}{(n-1)^2} \right]^2 = -\frac{1}{(n-1)^3} < 0$$

Hence, deleting a link in any complete network but the 2-line decreases one's network constraint.

Since the inequality for the other actor linked to the removed us the same and because other actors do not lose any ties, but are involved in less closed triads corollary 6 follows.

Corollary 5. The complete network is Pareto-dominated by the all-but-one-link network if $n > 2$.

Definition 12. A network is a *wheel* of size N if it is the union of a star of size N and a circle of size $N-1$ such that the circle spans the actors from group L of the star.

Theorem 7. No wheel with $n > 2$ is pair-wise stable.

Proof. See Appendix.

Theorem 8. No wheel with $n > 2$ is Pareto-optimal.

Proof. See Appendix.

Definition 13. A *cycle* network (Wasserman & Faust 1994: 171) is a connected network in which all actors have degree 2.

Theorem 9. The cycle network is pair-wise stable if and only if $n \in \{2, 4, 5\}$.

Proof. For $n > 5$, the path between some pair of actors is greater than 2. By theorem 1, pair-wise stability is then precluded. The triangle (the circle with $n = 3$) is not pair-wise stable, while the 2-Line ($n = 2$), the Box ($n = 4$), and the Pentagon ($n = 5$) are.

Theorem 10. The cycle network is Pareto-optimal if and only if $n \in \{2, 4\}$.

Proof. The cycle with $n = 5$ gives each actor a weakly higher network constraint than in the $2-(n-2)$ -complete bipartite graph. The 3-line gives each actor a lower network constraint than in the triangle. Cycle networks with $n = 2$ and $n = 4$ are complete bipartite graphs, and are Pareto-optimal by theorem 3.

Summarizing, this section has shown that the complete bipartite networks includes an important class of pair-wise stable networks and that the balanced complete bipartite networks are even unilaterally stable.

Some Examples

As mentioned before in the model section, strong stability implies unilateral stability, unilateral stability implies strong pair-wise stability, and strong pair-wise stability implies pair-wise stability. Here are three examples that illustrate the additional efficiency that is obtained through stronger stability requirements.

Figure 1 shows six pair-wise stable networks, of which we consider 3 here. All actors in the Octahedron are in structurally equivalent positions (Wasserman & Faust 1994: 356-357) and have a network constraint of $9/16$. This is clearly an inefficient network. Everyone is strictly better off in the balanced 6-actor complete bipartite network, where the network constraint is $1/3$. The Octahedron is pair-wise stable, but not strongly pair-wise stable. An actor does not wish to sever a single link: $c_i = 43/72 > 9/16$, but the deletion of precisely 2 links with unconnected neighbors does make her better off: $c_i = 1/4 < 9/16$.

The 4-star is Pareto-efficient by theorem 3, and the outer actors cannot do anything about the unequal division of wealth through strategic deletion of multiple links or addition of a single link. By theorem 10, however, one of the outer actors can make everyone but the middle actor

better off by deleting a tie with the middle actor and establishing two ties with the other two outer actors, thereby forming the 4-actor balanced complete bipartite network (or ‘Box’). The network is strongly pair-wise stable, but not unilaterally stable.

The Pentagon (or 5-cycle) is Pareto-deficient, but no network that is unilaterally obtainable from it makes the manipulator strictly better off. A coalition, however, can bring about a Pareto-improvement. For example, a coalition of four actors can obtain an balanced complete bipartite network by turning the 4-line that connects them into a Box.

FIGURE 1 ABOUT HERE

Simulation Results

In each simulation, we start from a given network. At each time one actor is chosen randomly. This actor will make that link change, addition or deletion, that leads to the greatest decrease in her Burt constraint and that – in the case of a link addition – does not make the other actor involved in the tie worse off. The simulation continues until no actor wants to change a tie anymore. By definition, this simulation ends in a pair-wise stable network.

We first ran such simulations on all 13,597 non-isomorphic networks with $n = 2$ to 8. In this way we must have identified all pair-wise stable networks for these network sizes, since whenever the initial network is pair-wise stable, it converges instantly.

In table 1, the number of stable networks is shown by network size and stability concept. For $n = 2$, we have the 2-line as the only pair-wise stable network. For $n = 3$, the 3-line is the only pair-wise stable network. For $n = 4$, the box is the only pair-wise stable network. For $n = 5$, there are two pair-wise stable networks, i.e., the Pentagon and the 2-3-complete bipartite network, of which only the latter is strongly stable. For $n = 6$, there are 4 pair-wise stable networks: the 2-4-complete bipartite and the 3-3-complete bipartite as well as the “bag” and the Octahedron. The Octahedron is not strongly pair-wise stable, and only the 3-3-complete bipartite network is unilaterally stable. For $n = 7$, there are 3 pair-wise stable networks: the 2-5-complete bipartite, the 3-4-complete bipartite, which is also unilaterally stable, and another network. For $n = 8$, there are 7 pair-wise stable networks: 2-6-complete bipartite, the 3-5-complete bipartite, the 4-4-complete bipartite, the “twisted cube” (see figure 1), the complete 4-partite network, and two other networks. The three densest networks, including the complete 4-partite network, are not strongly pair-wise stable. The “twisted cube” is the second unilaterally stable network for $n = 8$ in addition to the balanced complete bipartite network. Note that the twisted cube is in fact a very regular structure in which every one has three links and occupies a regularly equivalent position with all the others.

In addition, we checked for the more than 12 million non-isomorphic networks with $n = 9$ or 10 if they fulfilled a specific stability concept without simulating changes towards stability. In this way we found 9 pair-wise stable networks for $n = 9$ and 14 pair-wise stable networks for $n = 10$. The unilaterally stable structures for $n = 10$ are the balanced complete bipartite network and another very regular network in which every actor has four ties. Finally, for $n = 11 - 16$, we

checked if the networks that resulted from the simulations described below fulfilled stronger stability criteria than pair-wise stability. The results are also summarized in table 1. Of course for these network sizes, it is very likely that we did not find all pair-wise stable networks through the simulation.³ We know from the analytical results that some networks that we did not find in the simulations are nevertheless pair-wise stable. This is why for some network sizes, the lower bound of the number of pair-wise stable networks (table 1) exceeds the number of networks found in the simulations (table 2).

TABLE 1 ABOUT HERE

For $n = 9 - 25$, we drew an on density stratified sub-sample of networks. For densities for which the number of different networks is small, we use all networks, while we drew a random sample of connected networks in all other cases. We never let a simulation start from a disconnected network, but since disconnected networks cannot be pair-wise stable and since the minimal density of pair-wise stable networks seems to be around 0.4, not starting from the sparsest connected networks is not problematic. We decreased the number of networks per network density for larger network sizes, in order to have comparable numbers of networks per network size.⁴ In sum, in our construction of the set of initial networks, we attempted to minimize bias towards networks of a particular density while keeping the set feasibly small. For each network, we let the simulation twice converge to pair-wise stability twice. Convergence always occurred and it did so relatively fast, considering the network size. For $n = 25$, the maximum number of iterations to reach a pair-wise stable network was just above 200. Our sample is represented in table 2.

TABLE 2 ABOUT HERE

Examining the entire range from $n = 2$ to $n = 25$ in table 3, there are a number of important observations to be made. The number of pair-wise stable networks increases as n increases, although not entirely monotonically. Clearly, the number of pair-wise stable network is very small compared to the number of existing networks. Our qualitative observation in the previous section regarding the greater stability of balanced over unbalanced complete bipartite networks appears to be generalizable. The simulation clearly shows that the balanced complete bipartite graph is by far the most likely to emerge from the simulation. The more unbalanced a complete bipartite graph is, the less likely it emerges from the simulation. Table 3 shows the proportions of simulations from which the balanced and the least unbalanced complete bipartite networks emerge as pair-wise stable networks. These two networks cover about 85% of the resulting networks except for some cases in which n is small. If n is odd, the balanced complete bipartite network alone even accounts for 75-80% of the resulting pair-wise stable networks. There are no other pair-wise stable networks that occur in a large proportion of simulations except for the Pentagon (19%, all percentages are taken within network size) and the third pair-wise stable

³ Checking all structures for $n = 10$ took with our software and computers about 5 days, which would imply that for $n = 11$ it would take about 500 days.

⁴ Numbers sampled of networks per density were decreased at $n = 13$ and $n = 17$. As a result, there is a drop of sampled networks at $n = 13$ and $n = 17$ and an increase thereafter due to the increasing number of different densities if the network has more actors.

network for $n = 7$ (34%). For $n > 8$, no other network occurs in more than 8% of the simulations and for $n > 16$ in not more than 3%. Strikingly, the other unilaterally stable networks that are not bipartite do not emerge in larger percentages than other pair-wise stable networks. The twisted cube, for example, occurs in only 1% of the simulations for $n = 8$.

TABLE 3 ABOUT HERE

The density of pair-wise stable networks varies between 0.4 and 0.9. Another question is whether or not we can predict structure of the outcome network by the properties of the ingoing network. Because the dynamic process toward pair-wise stable networks has not much variation in terms of the outcome networks, we restrict ourselves here to the relationship between density of the ingoing network and the resulting network. Not surprisingly, denser starting networks lead to denser pair-wise stable networks. Furthermore, one can see from table 3 that the correlation between the two densities increases with network size.

It should be noted that the correlations as well as the proportions presented in table 2 are contingent upon the sample of networks that is used. However, the findings are so strong and the prevalence of the balanced complete bipartite networks is so obvious that we do not really worry about this. One thing we did was weighting networks for $n < 9$ with the inverse of the size of the isomorphism group that they were the representative of.⁵ The statistics then corresponds with a sample of random networks. This left the substantive results unaffected

Conclusion

In this paper, we have performed a thought experiment. The world we have modeled is one in which people follow Burt's advice to add ties only to those from which one does not already receive sufficient information through indirect channels, and delete ties only if they are redundant in terms of informational benefits. We have had actors behave under such a regime and have sought to answer the question: "What networks will evolve?"

In short, the answer is: Balanced complete bipartite networks. These networks meet all stability criteria from a recently emerged literature on dynamic networks and are Pareto-efficient. Moreover, in our simulations, most runs converged to such networks. In these networks, all actors are well-connected, none of these connections being redundant, and informational benefits are distributed equally.

Balanced complete bipartite networks have some interesting properties. First of all, no one really is a broker. Even though each actor attempts to occupy a brokering position, in these equilibria, 2-step information flow between any two actors travels through at least $\lfloor (n - 1) / 2 \rfloor$ third parties. Thus, betweenness centrality (Freeman 1974; Wasserman & Faust 1994: 189-191) is not particularly high for any single actor. Hence the conclusion "If everyone wants to be in the center, there is no center."

⁵ Sizes of automorphism groups were also determined using Nauty 2.2 (see McKay 1990).

Thus, although constraints in the most prevalent equilibrium networks are relatively low, actors hardly have any comparative advantage. One may therefore attempt to generalize our result and ask: “If the distribution of Burtian benefits were a zero-sum game, would in equilibrium everyone get about 0?” We ran models in which utility represented proportional information benefits. Results were very similar to the ones reported on in this paper, thereby answering the question affirmatively, because in these networks everybody has about the same network constraint.

It is kind of puzzling why the stable networks look so much as markets of, e.g., buyers and sellers, where it is better for the buyers that the sellers do not communicate and the other way round. But it is also better for sellers if they do not communicate among each other. It corresponds, however, with ideas on such markets that actors on the “short side” of the market are better off than those on the long side, which corresponds with that in unbalanced complete bipartite networks actors in the small group have a smaller constraint than actors in the large group. Of course, it is unlikely that a Burt’s model in which no “roles” in the network are identified would reflect phenomena in market models with different roles, although competition is an important aspect in Burt’s model.

While the dominant equilibrium type that we found resembles a market, it does not resemble a typical friendship network. We know that friendship networks tend to be small-world networks, i.e. various clusters connected through a few inter-cluster ties (Watts & Strogatz 1998). And we know something about the causes. People prefer to associate with people who are like them and therefore tend to know their contact’s contacts as well (Homans 1950; Newcombe 1961; Lott & Lott 1965; Byrne 1971; Cohen 1977; Kandel 1978; Verbrugge 1977). Segregated networks have been shown to be extremely stable (Carley 1991; Axelrod 1997; Mark 1998; Macy, Kitts, and Flache 2003). Burtian network dynamics are therefore somewhat difficult to imagine in non-business settings.

We finish with some remarks on possible extensions of our analysis. A seemingly obvious one would be adding explicit costs for maintaining ties because this is a common issue in economics articles on these kinds of dynamics. However, we do *not* think that this is an appropriate extension, because Burt’s network constraint implicitly takes costs of ties into account. If ties would be costless redundancy of information would hardly be an issue. One other assumption of the Burt constraint is that all ties to others are equally important. One could study a more flexible version of the Burt constraint in which actors can freely divide attention among their network relations. Finally and most importantly, it is unlikely that everyone in a network strives for structural holes and that everybody is equally able to structure her network in such a way that the network constraint is minimized. It would be interesting to see what would happen if one would not assume that all actors optimize network constraint, but instead that some have other aims such as optimizing local density or do not at all look at the structure of the network in terms of ties among others, but consider only relations within her egocentric network.

Appendix: Proofs of Theorems

Proof of Theorem 0. Recall that we can rewrite the constraint measure as:

$$c_i = \frac{1}{d_i^2} \sum_j \left[1 + \sum_k \frac{1}{d_k} \right]^2 \leq \frac{1}{d_i^2} \sum_j 2^2 \leq \frac{4}{d_i}$$

where j is the index for neighbors of i and k is the index for neighbors of i that are also connected to j . Thus, the theorem is true of $d_i \geq 4$. The remaining cases are $d_i = 2$ and $d_i = 3$. Here c_i reaches the highest value in the triad (complete three-actor network) in which every one has $c_i = 9/8$. These cases are easily enumerated because if neighbors of the focal actor have ties to actors that are not connected to the focal actor, this decreases the Burt constraint. This completes the proof.

Proof of Theorem 2. Removing a tie is not an option in a complete bipartite graph because that would create a shortest path longer than 2, and hence cannot be an improvement by theorem 1. Therefore, we only need to consider conditions under which group members create a tie within their group. The constraint in the complete bipartite network equals $\frac{1}{k}$ for the group of size l and $\frac{1}{l}$ for the group of size k . Creating a tie in the larger group of l actors changes the constraint of the two actors involved in that tie to

$$\frac{k}{(k+1)^2} \left[1 + \frac{1}{(k+1)} \right]^2 + \frac{1}{(k+1)^2} \left[1 + \frac{k}{l} \right]^2 = \frac{1}{(k+1)^2} \left[\frac{k(k+2)^2}{(k+1)^2} + \frac{(l+k)^2}{l^2} \right]$$

In order for the network to be pair-wise stable, this expression must be larger than $\frac{1}{k}$, or

$$\left[\frac{k(k+2)^2}{(k+1)^2} + \frac{(l+k)^2}{l^2} \right] - \frac{(k+1)^2}{k} > 0 \quad [1]$$

The same expression should hold for actors in the small group, but then with k en l reversed:

$$\left[\frac{l(l+2)^2}{(l+1)^2} + \frac{(l+k)^2}{k^2} \right] - \frac{(l+1)^2}{l} > 0 \quad [2]$$

Inequality [2] is satisfied for any $l \geq k \geq 1$, so it does not add an extra condition for a complete bipartite network to be a pair-wise stable. As a consequence, [1] is a necessary and sufficient condition, which is always fulfilled except for stars with $n > 4$. This completes the proof.

Proof of Theorem 3. Consider an actor i from the smaller group of $k \leq l$ actors. The network constraint of i can only be lower in another network if she has more than l ties. Let $a \leq k - 1$ be this additional number of ties of actor i , let j be the index for neighbors of i , and q the index for actors that i and j share as neighbors, and let π_j indicate the proportion of ties of j with other neighbors of i . Then, in order for i to have a lower network constraint, the following inequality must hold:

$$\begin{aligned} \frac{1}{l} &> \frac{1}{(l+a)^2} \sum_j \left[1 + \sum_q \frac{1}{d_q} \right]^2 \geq \frac{1}{(l+a)^2} \sum_j \left[1 + \sum_q \frac{1}{d_q} \right] = \frac{1}{(l+a)^2} \sum_j \left[1 + \sum_q \frac{1}{d_j} \right] \\ &= \frac{1}{(l+a)^2} \sum_j [1 + \pi_j] = \frac{1}{(l+a)} [1 + \bar{\pi}] \Rightarrow \pi_j < \frac{a}{l} \text{ for some } j. \end{aligned}$$

But for each j , in order to be at least as well off in the new network as in the complete bipartite network considered, her degree d_j must be at least k . Only $k - a - 1$ of j 's connections can be to actors that actor i is not connected to, thereby excluding i herself. For each j , π_j may therefore be no less than a over k :

$$\pi_j \geq \frac{d_j - k + a}{d_j} \geq \frac{a}{k} \text{ for all } j.$$

We have reached a contradiction. Thus, in order to improve i 's network constraint, at least one actor j must be given fewer than k neighbors and this actor is consequently strictly worse off in the new network than in the complete bipartite network considered.

A Pareto-improvement must therefore leave the network constraints of all actors with l ties unchanged and give them precisely l ties. This can only be done by keeping all ties that connect such an actor with an actor with l ties, or, in the case of $k = l$, switching to a network that contains a different k by l complete bipartite network. The only remaining possibility for a Pareto-improvement is therefore the addition of one or more ties between actors with k ties. But any such addition involves the creation of triangles and hence increases the network constraint of all actors with l ties. This renders the assumed Pareto-improvement impossible and completes the proof.

Proof of Theorem 4. *If.* Consider an actor i from the group of k actors. We know from theorem 3 that we cannot make this actor better off without letting one of her neighbors have a degree lower than k . But leaving $g \setminus h_{Nd(i,g)}$ unchanged, all her neighbors have at least degree k . In the even case, in which $k = l$, by symmetry, this impossibility of unilateral improvement extends to actors of the group of size l . The single remaining possibility for unilateral improvement is therefore a permitted decrement of the constraint of an actor i from the group of l actors in the uneven case, in which $k = l - 1$. Let $0 \leq b_k \leq k$ be the number of ties actor i has with actors from the group of size k in the new network, and let $0 \leq b_l \leq k$ be the number of ties she has with actors from the group of size l . Then, leaving $g \setminus h_{Nd(i,g)}$ unchanged, the following inequality must hold:

$$\frac{1}{k} > \frac{1}{(b_k + b_l)^2} \left(b_k \left[1 + \frac{b_l}{k+1} \right]^2 + b_l \left[1 + \frac{b_k}{k+1} \right]^2 \right) = \frac{b_k (b_l + k + 1)^2 + b_l (b_k + k + 1)^2}{(b_k + b_l)^2 (k + 1)} \Rightarrow$$

$$\begin{aligned} b_l^2 (k + 1)^2 + 2b_k b_l (k + 1) + b_k^2 (k + 1)^2 &> b_k k (k + 1)^2 + 2b_k b_l k (k + 1) + b_k b_l^2 k + b_l k (k + 1)^2 + 2b_k b_l k (k + 1) + b_l b_k^2 k \Rightarrow \\ b_l^2 (k + 1)^2 - 2b_k b_l (k - 1)(k + 1) + b_k^2 (k + 1)^2 &> b_k k (k + 1)^2 + b_k b_l^2 k + b_l k (k + 1)^2 + b_l b_k^2 k \Rightarrow \end{aligned}$$

$$[b_k(b_k - k) + b_l(b_l - k)](k+1)^2 > b_k b_l [(b_k + b_l)k + 2(k-1)(k+1)]$$

The left-hand side of this last inequality is never strictly positive, and the right-hand side is never strictly negative. Hence, it cannot be satisfied.

Only if. If $l - k > 1$, an actor from the larger group of l actors can delete all her ties with actors from the smaller group of k actors and add $l - 1$ ties to the other actors from the larger group of l actors. By doing so, she decreases her constraint from $\frac{1}{k}$ to $\frac{1}{l-1}$. By permitting this change the $l - 1$ actors see their constraint fall from $\frac{1}{k}$ to $\frac{1}{k+1}$.

Proof of Theorem 7. Denote n_2 a factor of n . Then for a complete n/n_2 -bipartite network with equal groups of n_2 size the following inequality should hold such that no one wants to sever a tie (note that we need only one equation because all actors have equivalent positions).

$$\begin{aligned} & \frac{1}{(n-n_2)^2} (n-n_2) \left[1 + \frac{n-2n_2}{n-n_2} \right]^2 - \frac{1}{(n-n_2-1)^2} \left[(n-2n_2) \left[1 + \frac{n-2n_2-1}{n-n_2} \right]^2 + (n_2-1) \left[1 + \frac{n-2n_2}{n-n_2} \right]^2 \right] < 0 \Leftrightarrow \\ & \frac{(2n-3n_2)^2}{(n-n_2)^3} - \frac{1}{(n-n_2-1)^2} \left[\frac{(n-2n_2)(2n-3n_2-1)^2 + (n_2-1)(2n-3n_2)^2}{(n-n_2)^2} \right] < 0 \Leftrightarrow \\ & \frac{(2n-3n_2)^2}{(n-n_2)^3} - \frac{1}{(n-n_2-1)^2} \left[\frac{(n-n_2-1)(2n-3n_2)^2 - (n-2n_2)(4n-6n_2-1)}{(n-n_2)^2} \right] < 0 \Leftrightarrow \\ & \frac{(n-2n_2)(4n-6n_2-1)}{(n-n_2-1)^2 (n-n_2)^2} - \frac{(2n-3n_2)^2}{(n-n_2-1)(n-n_2)^3} < 0 \Leftrightarrow \\ & (n-2n_2)(4n-6n_2-1)(n-n_2) - (n-n_2-1)(2n-3n_2)^2 < 0 \Leftrightarrow \\ & 4(n-2n_2)(n-n_2)(n-1.5n_2-0.25) - 4(n-n_2-1)(n-1.5n_2)^2 < 0 \Leftrightarrow \end{aligned}$$

which is always true because

$$(n-2n_2)(n-n_2) < (n-1.5n_2)^2 \text{ and } (n-1.5n_2-0.25) < (n-n_2-1) \text{ if } n_2 \geq 2.$$

In order for no actor to benefit from adding any of her structurally equivalent potential links in this complete n/n_2 -bipartite network with equal groups, the following inequality must hold:

$$\begin{aligned} & \frac{1}{(n-n_2)^2} (n-n_2) \left[1 + \frac{n-2n_2}{n-n_2} \right]^2 - \frac{1}{(n-n_2+1)^2} \left[(n-n_2) \left[1 + \frac{1}{n-n_2+1} + \frac{n-2n_2}{n-n_2} \right]^2 + \left[1 + \frac{n-n_2}{n-n_2} \right]^2 \right] < 0 \Leftrightarrow \\ & \frac{(2n-3n_2)^2}{(n-n_2)^3} - \frac{[(n-n_2+1)(2n-3n_2) + (n-n_2)]^2 + 4(n-n_2+1)^2(n-n_2)}{(n-n_2+1)^4(n-n_2)} < 0 \Leftrightarrow \\ & (2n-3n_2)^2(n-n_2+1)^4 - [(n-n_2+1)(2n-3n_2) + (n-n_2)]^2 + 4(n-n_2+1)^2(n-n_2)(n-n_2)^2 < 0 \Leftrightarrow \\ & x^4(7-6n_2) + x^3(12-18n_2) + x^2(4-16n_2) - 4xn_2 - 2x^4n_2^2 + 2x^3n_2^2 + 5x^2n_2^2 + 4xn_2^2 + n_2^2 < 0, \text{ where } x = n-n_2. \text{ Since } n_2 < x = n-n_2, \text{ the equation above is implied by (replacing } n_2^2 \text{ by } xn_2) \end{aligned}$$

$$x^4(7-4n_2) + x^3(12-13n_2) + x^2(4-12n_2) - 3xn_2 < 0,$$

which is obviously true, because $n_2 \geq 2$ and $x > 0$. This completes the proof.

Proof of Theorem 8. By theorem 4, the triangle (= the wheel with $n = 3$) is not pair-wise stable. A necessary condition for pair-wise stability of wheels is that the actors of the group with each l ties cannot decrease their network constraints by deleting a link with the central actor. For $n > 3$, this condition is represented by the following inequality:

$$\frac{1}{2} > \frac{1}{3^2} \left[\left(1 + \frac{2}{3}\right)^2 + 2 \left(1 + \frac{1}{n-1}\right)^2 \right] \Rightarrow n < 0, \text{ a contradiction. This completes the proof.}$$

Proof of Theorem 9. The 3-line gives each actor a lower network constraint than the triangle (= the wheel with $n = 3$). A necessary condition for a wheel with $n > 3$ to be Pareto-optimal is that at least one actor has a strictly lower network constraint in the wheel than in the 3, $n-3$ complete bipartite network or all actors have the same network constraints in both networks. The l actors all have higher constraints in the wheel than in the 3, $n-3$ complete bipartite network, because they have degree 3 in both cases but are only involved in triangles in the wheel. Therefore, the central actor must be worse off in the 3, n complete bipartite network:

$$\frac{1}{n-3} > \frac{1}{(n-1)^2} (n-1) \left(1 + \frac{2}{3}\right)^2 \Rightarrow n < 5$$

The only remaining wheel with $n = 4$ is the complete 4-actor network. In this network everyone has a higher constraint than in the 2, 2 complete bipartite network. This completes the proof.

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Table 1: Number of networks (found) for various stability criteria (note that we do not have any guarantee that we have all pair-wise stable networks for networks with $n > 10$).

n	# Non-isomorphic graphs	Connected	Deletion proof	Strongly deletion proof	Pair-wise stable	Strongly pair-wise stable	Unilaterally stable	Strongly stable
2	2	1	1	1	1	1	1	1
3	4	2	1	1	1	1	1	1
4	11	6	3	3	2	2	1	1
5	34	21	7	6	2	2	2	1
6	156	112	21	19	4	3	1	1
7	1044	853	68	62	3	3	1	1
8	12346	11117	304	286	10	7	2	1
9	274668	261080	1841	1733	9	7	1	1
10	12.01e6	11.72e6	18972	17795	14	9	2	1
11	10.19e8	10.07e8	?	?	≥ 13	≥ 8	≥ 1	≥ 1
12	16.51e10	16.41e10	?	?	≥ 25	≥ 12	≥ 1	≥ 1
13	50.50e12	50.34e12	?	?	≥ 17	≥ 8	≥ 1	≥ 1
14	29.05e15	29.00e15	?	?	≥ 21	≥ 10	≥ 1	≥ 1
15	31.43e18	31.40e18	?	?	≥ 21	≥ 10	≥ 1	≥ 1
16	64.00e21	63.97e21	?	?	≥ 34	≥ 16	≥ 2	≥ 1
					≥ 43	≥ 21	≥ 2	≥ 1

Table 2: Description of the sample of initial networks in the simulation.

- All non-isomorphic networks for $n = 2 - 8$ ($N = 13,597$)
- Selection for n 9-12: (sampled: $N = 29,300$); (other $N = 12,690$)
 - Size 9: all connected with 8, 9, 10, 26-36 ties; 400 for 11-25 ties
 - Size 10: all connected with 9, 10, 36-45 ties; 300 for 11-35 ties
 - Size 11: all connected 10, 11, 46-55 ties; 200 for 12-45 ties
 - Size 12: all connected 11, 57-66 ties; 200 for 12-56 ties
- Selection for n 13-16: (sampled 15,200); (others 8,998)
 - Size 13: all for 69-78 [2,228]
 - Size 14: all for 82-91 [2,250]
 - Size 15: all for 96-105 [2,258]
 - Size 16: all for 111-120 [2,262]
 - Size 13-16: 50 for $N-1$ to $N*(N-1)/2-10$ ties
- Selection for size 17-25: (sampled 33,900); (other $N=414$)
 - all for $N*(N-1)/2$ to $N*(N-1)/2-5$, [$9 * (1 + 1 + 2 + 5 + 11 + 26) = 414$]
 - 20 for $N-1$ to $N(N-1)-6$

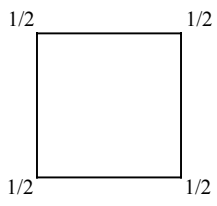
Total: $13,597 + 29,300 + 12,690 + 15,200 + 8,998 + 33,900 + 414 = 113,299$

Table 3: Simulation results*

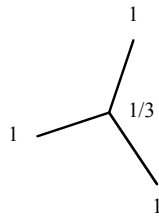
N	# from simulations	from theorem m -partite except $n = 10$	Full bipartite, ($\lfloor n/2 \rfloor, \lceil n/2 \rceil$)	Full bipartite ($\lfloor n/2 \rfloor - 1, \lceil n/2 \rceil + 1$)	Other prominent pairwise stable	Other pairwise stable
2	1	0	1	n.a.	n.a.	n.a.
3	1	0	1	n.a.	n.a.	n.a.
4	2	0	.86	.14	n.a.	n.a.
5	2	0	.81	n.a.	.19 (T2: 5, US)	n.a.
6	4	0	.71	.18	.09 (T2: 4, T3: 2)	.02 (T4: 6)
7	3	0	.63	.03	.34 (T2: 1, T3: 6)	n.a.
8	10	0	.62	.15	.10 (T3: 6, T4: 2) .11 (T2: 1, T3: 4, T4: 3) .01 (T3: 8, US)	.01
9	9	0	.87	.01	.08 (T3: 2, T4: 7)	.04
10	11	(3 checking all)	.69	.23	.04 (T3: 2, T4: 4, T5: 4) .02 (T4: 10, US)	.02
11	12	1	.88	.04	.03 (T4: 7, T5: 4)	.05
12	23	2	.61	.31	.04 (T4: 2, T5: 10)	.04
13	15	2	.81	.05	.07 (T8: 5, T9: 8)	.07
14	18	3	.54	.30	.07 (T9: 10, T10 = 4)	.09
15	31	3	.79	.06	.04 (T10: 15) .01 (T6: 15, US)	.10
16	39	4	.51	.30	.06 (T10: 6, T11: 10) .0001 (T6: 16, US)	.13
17	35	4	.79	.06	n.a.	.15
18	42	6	.53	.32	n.a.	.15
19	44	5	.79	.07	n.a.	.14
20	56	6	.52	.34	n.a.	.14
21	59	7	.77	.08	n.a.	.15
22	73	7	.50	.35	n.a.	.15
23	86	6	.75	.09	n.a.	.16
24	107	8	.50	.33	n.a.	.17
25	110	8	.74	.10	n.a.	.16

* Tx: y means that in the stable networks there are y actors with x ties. US = unilateral stable.

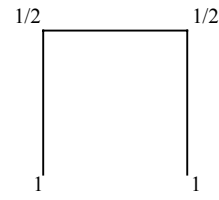
Figure 1: The Box, 4-star, 4-line, Pentagon, Twisted cube, and Octahedron. Numbers are constraint values.



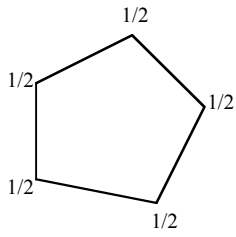
Box



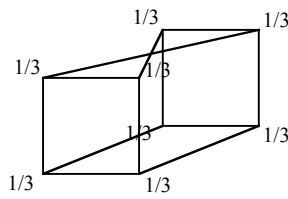
4-star



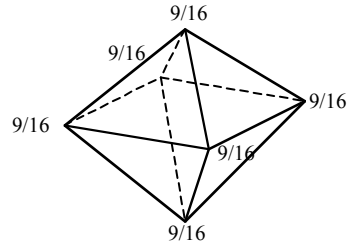
4-line



Pentagon



Twisted cube



Octahedron