Coalition Formation and Exclusion from a Common Pool Resource.

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> The tragedy of the commons develops in this way. Picture a pasture open to all. It is to be expected that each herdsman will try to keep as many cattle as possible on the commons. Such an arrangement may work reasonably satisfactorily for centuries because tribal wars, and disease keep the numbers of man well below the carrying capacity of the land. Finally, however, comes the day [...] when the long-desired goal of social stability becomes a reality. At this point, the inherent logic of the commons remorselessly generates tragedy.
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> Garrett Hardin (1968), *The tragedy of the Commons.*

> Suppose a society to fall into such want of all common necessaries that the utmost frugality and industry cannot preserve the greater number from perishing, and the whole from extreme misery: It will readily be admitted that the strict laws of justice are suspended in such pressing emergence, and give place to the motives of necessity and self-preservation..."
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> David Hume (1751), *An Enquiry Concerning the Principles of Morals.*

1 The Model

1.1 Basic notation and assumptions

Consider a set $N = \{1, 2, ..., n\}$ of players that try to gain the access or control over a common pool resource by excluding the rest of individuals. However, they are constrained in their decisions: Each of them possesses one unit of initial endowment that can be transformed into effort in the exclusion contest or in labor (the labor choice has effect only if the agent finally obtains the control of the resource). We denote these investments by r_i and l_i respectively, under the constraint that $r_i + l_i = 1$. Players are identical in the sense that they have the same endowments and face the same rate of transformation between the two activities.

Furthermore, we accept the possibility that players may be interested in forming coalitions in order to get a better chance of acceding to the common good. Let $\mathcal{N}$ denote the set of all nonempty coalitions. For each coalition $S \in \mathcal{N}$, let us denote its cardinality by s. A *coalition structure* π is a collection of disjoint coalitions $\{S_k\}_{k\in K}$ whose union is N (i.e., a partition of N). Let Π denote the set of all coalition structures.

Given a coalition structure π, let us denote by $\mathbf{r}(\pi) = (r_{S_1}, r_{S_2}, ..., r_{S_k})$ the vector of coalitional investments in excluding the outsiders. The result of the contest among coalitions is driven by the *conflict technology* (or *Contest Success Function)* that maps $\mathbf{r}(\pi)$ to a vector $\mathbf{p} = \{p^{S_k}\}_{k\in K}$ of coalitional winning probabilities (with probability p^S the coalition S controls the resource and so on). Of course, the sum of probabilities across elements of a given coalition structure adds up to 1.

We employ the well-known ratio form for the conflict technology, that characterizes the coalitional winning probability as proportional to $(r^S)^m$, where r^S is the sum of individual efforts of its members, and the exponent m represents the returns to scale or *effectivity* of conflict effort. Hence, coalition S gains the access to the resource with probability

$$p^S(\mathbf{r}) = \frac{(r^S)^m}{(r^S)^m + r^{-S}} \tag{1}$$

where $r^{-S} = \sum_{S_k \in \pi \setminus \{S\}} (r^{S_k})^m$ provided that coalition structure π has formed.

This form captures two extreme cases, $m = 0$ and $m = \infty$. The former is an equal chance lottery among the elements of π; the latter is just a first-price auction where the coalition with the highest aggregate effort wins the contest with probability 1.

Once a coalition has acceded to the resource, their members exploit it **separately.** Production is carried through the function $f(l^S)$, where $l^S = \sum_{i\in S} l_i$, that transforms the total labor input into output. This function is assumed to satisfy $f(0) = 0$ and to be increasing and concave in labor (otherwise there would not be 'tragedy of the commons'); labor has a constant unit cost of c. We also require that $f'(0) > c$ in order to ensure existence of interior solutions to the exploitation problem. It will be also important for our analysis to define the elasticity of production with respect to labor

$$\varepsilon = \frac{f'(l^S)l^S}{f(l^S)}$$

because we will use it as a proxy of scarcity: With concave technologies this elasticity is always smaller than one. Finally, let us define a partial ordering on the set of possible concave technologies: Technology f is said to *dominate* technology g if and only if $\varepsilon_f > \varepsilon_g$ for any level l of total labor input.

1.2 Description of the game

By the very nature of common goods, the more the players who exploit them in a noncooperative manner, the lower the individual payoff and the more severe the efficiency problems due to overexploitation. Because of that, we consider the possibility of agents fighting in an exclusion contest in order to gain the access and control of the common pool resource. However if agents expect that their probability of winning the contest is low, they may find worthy to join with other players: This may increase their probability of success at the cost of diminishing their share of the final "pie".

- *Stage 1:* Players form coalitions (let us postpone this issue to Section 3) and decide *individually* (in order to maximize individual expected payoffs) and simultaneously how much effort they devote to the success of the coalition in the exclusion contest. Hence, the strategy for a player $i \in S$ is a level of effort $r_i \in [0, 1]$; this choice yields in aggregate a coalitional effort r^S. Then, nature moves and with probability p^S, coalition S accedes to the second stage. With probability $1 - p^S$ it is excluded and the continuation payoff is simply zero.

  If players form the grand coalition, all of them accede to the resource without any contest.

- *Stage 2:* The result of the conflict is realized, that is, a coalition S wins, accedes to the resource and its members exploit it *separately* by using their remaining endowments. Then, a member of the winning coalition has to choose the amount of labor $l_i \in [0, 1 - r_i]$ he wants to put in order to exploit the resource. The payoff in this stage is therefore the *unique* Nash Equilibrium of the standard 'tragedy of the commons' under a restricted strategy space. Let us denote by l_i^* the equilibrium strategy. Then, the total production $f(L^*)$ depends solely on the total amount of labor input $L^* = \sum_{i \in S} l_i^*$ and the individual payoff in this stage is the individual share of total output minus the total cost of the labor choice[1], i.e. $\frac{l_i^*}{L^*} f(L^*) - cl_i^*$.

[1] This formulation is a feature of the common goods models. They tipically assume this form for the individual share. With respect to the unit cost of labor formulation, it can be found, among others, in Cornes and Sandler (1983) and Meinhardt (1999); we adopt it in order to avoid the labor-leisure trade-off, as for instance in Roemer (1996), and to focus just on the trade off between exclusionary and productive activities.

If the grand coalition forms, the total labor input is $l^*(N)$ and by symmetry the payoff is simply:

$$u_i^N = \frac{1}{N}[f(l^*(N)) - l^*(N)c].$$

If not, the expected payoff of player i when he is a member of coalition S given his optimal choices at each stage of the game is

$$u_i^S = \frac{(r^S)^m}{(r^S)^m + r^{-S}}[\frac{l_i^*}{L^*}f(L^*) - cl_i^*] \tag{2}$$

because if coalition does not win the contest it receives zero. And the joint payoff of the coalition is just

$$\sum_{i \in S} u_i^S = \frac{(r^S)^m}{(r^S)^m + r^{-S}}[f(L^*) - cL^*]$$

that clearly depends not only on the individual decisions of its members but also on the effort levels chosen by the other players invest in exclusion because (a) fixed a level of coalitional effort, if other appropriators decide to increase their effort, winning probability decreases (b) if coalitions play best responses, a change in r^{-S} both affects the choice of effort r_i and modifies the strategy space at stage 2.

1.3 Analysis of the game

Analysis of stage 2: At this point, members of the winning coalition exploit the common resource with the endowments left. Let us denote by r^* the previous level of effort (given that players are identical it is the same for all members). How much they will spend in production?

When s symmetric players exploit a common pool resource separately without any previous contest, it can be easily shown[2] that the total amount of labor invested l^S satisfies

$$\frac{1}{s}f'(l^S) + \frac{s-1}{s}\frac{f(l^S)}{l^S} = c \tag{3}$$

This means that the total labor input is a weighted average between the efficiency level that makes $f'(l_i) = c$ (achieved when only one agent entries) and the equalization to the average productivity. Due to symmetry we can denote this level simply by $l^*(s)$. Moreover, the Nash equilibrium total amount of labor is increasing in s and payoffs are decreasing (this is the reason why players may want to exclude each other). Then, the optimal

[2]See for instance Cornes and Sandler (1983) or Funaki and Yamato (1999).

labor choice l^* in the second stage, provided that a coalition s has won the contest and spent r^* in exclusion, is simply:

$$l^* = \begin{cases} l^*(s)/s & \text{if } 1-r^* \geq l^*(s)/s \\ 1-r^* & \text{if } 1-r^* < l^*(s)/s \end{cases} \quad (4)$$

Analysis of Stage 1: If players form the grand coalition there is no contest; for any coalition $S \subset N$, the level of effort per member, r_i, devoted to exclusion is privately determined.

Notice that players will want to use their entire endowments in both activities. Suppose a given choice of effort and labor that does not sum to one. Then, given that winning probability is increasing in own effort, expected utility increases if he invests an epsilon plus in exclusion. So, one can rewrite expression (2) and obtain

$$u_i^S(r^S, r^{-S}) = \frac{(r^S)^m}{(r^S)^m + r^{-S}}\left[\frac{1-r_i}{s-r^S} f(s-r^S) - c(1-r_i)\right]. \quad (5)$$

Hence, players' choice of effort follows the maximization of (5) subject to $0 \leq r_i \leq 1$.

Inspection of (5) reveals that members efforts are strategic substitutes, coalitional winning probability is a sort of public good. Moreover, the share of final production in case of victory decreases with effort. It is evident that both features generate free-riding incentives. However, as we will see later, this effect will be limited to small coalitions.

Let us now introduce the notion of best reply of an agent:

Definition 1 (Best Reply) *Let us denote by $r^S \backslash r_i = (r_1, ..., r_i', ...r_s)$ the strategy profile after the unilateral deviation of player i from the strategy profile $r^S = (r_1, ..., r_i, ..., r_s)$ Then, denote by $B_i^S(r_{-i})$ the set of best replies of agent i who is member of a coalition S to the strategies $r_{-i} = \{r_j\}_{i \neq j}$, chosen by his partners in S (if any) and the outsiders:*

$$B_i^S(r_{-i}) = \left\{r_i \in [0,1] \ / \ u_i^S(r^S, r^{-S}) \geq u_i^S(r^S \backslash r_i, r^{-S})\right\}$$

However, not all efforts are good candidates to best replies: It can be easily shown that, as a consequence of both symmetry and the future play in Stage 2, we can delete an interval of strictly dominated strategies:

Lemma 2 *No agent belonging to a coalition of size s will put less effort than $1 - \frac{l^*(s)}{s}$ nor 1.*

Proof. On the contrary, suppose that the at $1 - \frac{l^*(s)}{s}$ this agent decides to withdraw a ϵ of effort on labor. The remaining labor input is now $\frac{l^*(s)}{s} + \epsilon$. The first consequence is that coalitional winning probability decreases. Moreover, if he employes this ϵ as labor input, the increase in both total production and in his share do not compensate its cost (by the optimality condition, further increments of labor put beyond l*(s) have this effect). Hence, his individual payoff would go down for sure. The second part of the statement is straightforward. ■

This sets a ground floor of effort $r^{\min}(s) = 1 - \frac{l^*(s)}{s}$ which will be positive whenever optimal solutions to the production stage were interior. This will avoid further problems with ratio class of CSF's[3].

For our purposes, it will be essential to express individual payoffs (and coalitional too) with respect to the effort exerted by the rest of players. Then, we can define an indirect payoff function by supposing that $r_i(r^{-S})$ is a maximizer of $u_i^S(r^S, r^{-S})$ (with some abuse of notation, because agents take as given their peers' choices too) and taking into account that best responses must be identical for members of S (i.e., $r^S(r^{-S}) = sr_i(r^{-S})$)

Definition 3 *Let $r_i(r^{-S}) \in B_i^S(r_{-i})$ be the maximizer of the individual payoff of a member i of coalition S. Then, we define the* ***indirect payoff function*** *as*

$$u_i^*(r^{-S}) = u_i^S(sr(r^{-S}), r^{-S}) = \underset{r_i}{Max}\ u_i^S(r^S, r^{-S}) \tag{6}$$

2 First results

2.1 Best reply and reaction function

We already know that the optimal choice will be between $r(s)^{\min}$ and 1. This lower bound will vary according to the size of the coalition, the cost of labor and the technology:

Lemma 4 *The lower bound of exclusion effort $r(s)^{\min}$ for a member of coalition S*

(i) is increasing in size for any $s \geq 2$,

(ii) given two production functions f and g, such that f dominates g, $r_g(s)^{\min} > r_f(s)^{\min}$ for any s.

[3]This kind of functions presents inconsistency problems when $r_i = 0$ for all players. See Neary (1997) for a nice exposition of these questions.

Proof. See the Appendix. ∎

Point (i) shows that the 'lower-bound' effect will make free-riding problems disappear: We will see later that from a given a critical mass and on, individual effort contributions are increasing in size.

The part (ii) of the Lemma is also important for our investigations: If elasticity of labor is low (more concave technology, loosely speaking) the opportunity cost of exclusion activities is small we should expect greater levels of conflict than under nearly linear technology (and closer to 'abundance' then).

However, Lemma 4 does not actually inform us about the final choice of players. We do not know how much players will depart from the lower bound. This is because we have not considered yet the other main ingredient of our model: The conflict technology. Now, we can complete the characterization of the optimal effort choice through solving the individual maximization program:

Lemma 5 *In the Stage 1 of the Exclusion game, the* ***unique*** *interior best reply* r^* *by an individual member of a coalitional* S*, given the strategy* r^{-S} *of their rivals and the effort of the rest of his partners is given by*

$$\frac{m}{sr}\frac{r^{-S}}{(sr^*)^m + r^{-S}}[f(s-sr^*) - c(s-sr^*)] = f'(s-sr^*) + (s-1)\frac{f(s-sr^*)}{s-sr^*} - sc \tag{7}$$

Proof. See the Appendix. ∎

The effort choice (i.e., the gap with respect to lower bound) is increasing in outsiders' effort, as formally stated by the following lemma:

Lemma 6 *The best reply* $r_i(r^{-S})$ *is strictly increasing and concave in* r^{-S}.

Proof. See the appendix. ∎

This result shows that common-pool resources induce indeed exclusion races. Players always respond with more effort to an increasing in hostilities by outsiders. What drives the result is the 'no regret' nature of effort investment: Although, the value of the prize is endogenous (a distinctive characteristic of conflict models opposite to rent-seeking models[4]), players do not pay anything in case of loosing the contest.

[4]We refer the reader again to Neary (1997) for an exposition of the differences between these two kinds of models of rivalry.

2.2 The effect of technologies and size

Our target now is to investigate the effect of different productive and conflict technologies, parametrized by ε and m respectively, and of coalitional size on the agents' optimal choices. But let us first introduce a very important characterization of the optimal choice: when a coalition overexploits the resource.

One can know it by solving the difference $f(s - sr^*) - c$ in (7). Then, the coalition S overexploits the resource if

$$s > m(1 - p^S)\frac{1 - r^*}{r^*} + 1$$

This expression illustrates most of the effects underlying the game: If m is high, equalization of marginal productivities implies a more investment in conflict, that avoids excessive labor investment. The ratio $\frac{1-r}{r}$ increases as the technology becomes more linear, alleviating the negative consequences of overexploitation. Finally, bigger coalitions are more likely to overexploit because they are more attached to $r^{\min}(s)$ (that is increasing in size) that smaller ones. Notice that if external threat is strong enough conflict turns out to be a discipline device that alleviates the 'tragedy of the commons'.

2.2.1 Scarcity and conflict effectiveness

By the discussion on the lower bound of conflict effort, we should expect that the optimal choice of a representative member of coalition S must be increasing in our proxy of scarcity, $1 - \varepsilon$. The interpretation is simple: For low elasticity technologies, the marginal productivity of productive activities is likely to fall below marginal returns of exclusion. Moreover, free-riding becomes less attractive because diverting endowments to production pay less.

Lemma 7 *Given two production functions f and g such that f dominates g the individually optimal effort choice of agent $i \in S$ under g is higher than under f ceteris paribus.*

Proof. We apply the implicit function theorem. In this case $h(s, m, \varepsilon) = \frac{mr^{-S}}{(r^S)^m + r^{-S}}\frac{1}{s}[f(s - r^S) - c(s - r^S)] - \frac{r^S}{s}[(s + \varepsilon - 1)\frac{f(s-r^S)}{s-r^S} - sc]$ and then $\frac{\partial r_i(\varepsilon)}{\partial \varepsilon} = \frac{-\partial h/\partial \varepsilon}{\partial h/\partial r_i}$. By concavity we know that $\frac{\partial h}{\partial r_i} < 0$, and $\frac{-\partial h}{\partial \varepsilon} = \frac{f(s-r^S)}{s-r^S} > 0$. Hence, individual effort is decreasing in elasticity. ■

This shows again that conflict is definitively linked with scarcity, at least in the way we have modelled it, by the elasticity of the production function: In a world of abundance (under linear technology) overexploitation does not exist. In such case, conflict activities make less sense. But, when others' actions reduce severely your payoff possibilities, agents can take advantage of exclusion effort.

The effectivity of conflict provides us the other key for our investigations: The greater the value of m the bigger the conflict effort we should observe. This particularly makes sense when marginal productivity tends to be low (due to a high degree of concavity) and conflict is very effective: By equalization of marginal returns of both investments, players will put much more effort in excluding the others. Casual observations confirm that conflict is higher in poor but relatively high armed societies.

Unfortunately, results we get are not as clear as for the previous case. We can extract only partial conclusions.

Lemma 8 *Ceteris paribus, the optimal effort choice for an agent $i \in S$ is increasing in m, if the coalition structure consist of coalitions of the same size or r^S is "sufficiently" small.*

Proof. Rearranging:

$$\frac{m}{r^S}\frac{r^{-S}}{(r^S)^m + r^{-S}}[f(s-r^S) - c(s-r^S)] - [(s-1)\frac{f(s-r^S)}{s-r^S} + f'(s-r^S) - sc] = 0$$

Recall first that $r^{-S} = \sum_{k\in\pi\setminus S}(r^{S_k})^m$. Then, by the implicit function theorem and given that $\frac{\partial r_i(m)}{\partial m} = \frac{-\partial h/\partial m}{\partial h/\partial r_i}$ then the relationship between best reply and conflict effectiveness is driven by the sign of

$$-\frac{\partial h}{\partial m} = -[f(s-r^S)-c(s-r^S)][\frac{r^{-S}}{(r^S)^{m+1}+r^{-S}r^S} + \frac{m}{r^S}\frac{(r^S)^m\sum_{k\in\pi\setminus S}(r^{S_k})^m(\ln r^{S_k} - \ln r^S)}{((r^S)^m+r^{-S})^2}]$$
$$= -\frac{[f(s-r^S)-c(s-r^S)]}{r^S((r^S)^m+r^{-S})}[\sum_{k\in\pi\setminus S}(r^{S_k})^m(1+mp^S(\ln\frac{r^{S_k}}{r^S}))]$$

When the coalition structure is symmetric, the efforts put by coalitions are the same, so the differences of logarithms is zero. Then $\frac{\partial r_i(m)}{\partial m} > 0$. On the other side, if coalition S is small relatively to the other elements of π, (and its effort choice is not too high) or conflict effort is low, $\ln r^S$ is very negative, the logarithm is positive and we can be sure about the sign of the derivative. Effectiveness amplifies these effects. Out of these cases, the sign is ambiguous. ■

We can only proof the intuitive result that the more efficient the conflict the more the conflict observed at some very particular cases.

2.2.2 Size and free riding

So far, we know two effects of size in agents' optimal choices: First, winning probability is a public good that would induce free-riding as size increases. Second, we know by Lemma 4-(i), that the level $l^*(s)/s$ is decreasing for $s \geq 2$ and therefore that the lower bound of conflict effort increases with size. Expression (6) tells us that two opposite forces are working at the same time: Free-riding at the winning probability side and size power at the production part.

As the following Lemma shows from a threshold size s^* and on this effect will push upwards the effort contribution, avoiding a systematic free-riding on coalitional probabilities. Therefore, for sufficiently large coalitions, the lower bound effect will dominate the free-riding incentives.

Lemma 9 *There exists a threshold size s^* such that for any $s \geq s^*$ the individual optimal choice of effort is increasing in size.*

Proof. See the Appendix. ∎

2.2.3 An Example

Let us illustrate the results above with the following example. Take $N = 3$ and suppose initially that the technology of production $f(l) = l^{0.4}$, the CSF takes the form (1) with $m = 1$ and the cost of labor $c = 0.5$. Given that the singleton coalition structure and the grand coalition are trivial, we focus on the case when $\pi = \{\{i, j\}, \{k\}\}$

In Figure 1 we plot the effect of different technologies, both of production and exclusion, on the best reply effort levels of members of $\{i, j\}$ (vertical axis) and of $\{k\}$ (horizontal one) given the outsiders' choices. The intersection of the reaction functions constitute the Nash Equilibrium of the game.

In panel 1a, the dashed lines are the reaction functions when technology is $f(l) = l^{0.8}$. As stated above, players have less incentives to put effort given that "scarcity" or rivalry in production has been alleviated. Notice that because the higher value of the final "prize", the reaction function for player k is steeper.

Insert Figures 1a and 1b here (8)

In panel 1b, the dashed lines correspond to the case when $m = 2$. As pointed out in Lemma 9, the effect of this change is ambiguous: For low values of outsiders' effort, best replies lie below. However, the equilibrium takes place at higher investments in exclusion.

Finally, the effect of size on individual effort is plotted (once fixed a level of outsiders' effort) in Figure 2. The dashed lines represent again the case when $f(l) = l^{0.8}$. In each case, the upper line is the optimal individual choice whereas the line below is the lower-bound of conflict effort described by expression (3).

$$\textbf{Insert Figure 2 here} \tag{9}$$

As stated in Lemma 8, investments in exclusion are higher under the dominated technology for any coalitional size. The other main feature is that, beyond the threshold size s^*, around 2, individual effort converges to $r(s)^{\min}$ as size increases.

3 Coalition formation

3.1 Static approach

The main difficulty of the non-orthogonal games of coalition formation, in contrast with standard characteristic form games, is that outsiders' actions affect coalitional payoffs. Static approaches simplify this issue by assuming some pattern of behavior for the rest of players that pins down a coalitional payoff. This is the origin of the concepts of alpha and beta characteristic form introduced by Aumann (1959). By symmetry, we can characterize these concepts just by the individual payoff instead that by a profile of them.

Definition 10 *The α-characteristic function, v_α, in the exclusion game is defined by:*

$$v_\alpha(S) = \underset{r_i}{Max}\ \underset{r^{-S}}{Min}\ u_i^S(r^S, r^{-S}) = \underset{r_i}{Max}\ u_i^S(r^S, \widehat{r}^{-S}) = u_i^S(r^S(\widehat{r}^{-S}), \widehat{r}^{-S}) = u_i^*(\widehat{r}^{-S}) \tag{10}$$

where $\widehat{r}^{-S}$ is the minimizer. So, this expression corresponds to the indirect payoff function (6) when the outsiders have chosen the action (and therefore a partition) that minimizes the coalitional payoff: It is the minimum payoff that members of S can guarantee to themselves.

The beta notion defines the payoff members cannot prevented from for any choice of outsiders:

Definition 11 *The β-characteristic function v_β in the exclusion game is defined by:*

$$v_\beta(S) = \underset{r^{-S}}{Min}\ \underset{r_i}{Max}\ u_i^S(r^S, r^{-S}) = \underset{r^{-S}}{Min}\ u_i^S(r^S(r^{-S}), r^{-S}) = \underset{r^{-S}}{Min}\ u_i^*(r^{-S}) \tag{11}$$

Notice that both characteristics forms will coincide if $\widehat{r}^{-S} = Min_{r^{-S}}\ u_i^*(r^{-S})$. Unfortunately we cannot ensure that this is always the case.

Lemma 12 *The indirect payoff function of a player $i \in S$ is strictly decreasing in r^{-S} if $s < (m+1)(1-p^S)\frac{l}{r} + 1$.*

Proof. See the Appendix. ■

Why the indeterminacy on the effect of r^{-S}? Notice first that if a coalition is underexploiting ($f'(l^S) < c$) the sign of the derivative is negative for sure. This can be easily observed by comparing the threshold above and the threshold of overexploitation **??**. However this is not necessarily true when the coalition is overexploiting the resource. In such cases, conflict works as a discipline device that deters agents of spending too much in exploiting the resource. Then, by reducing their labor input agents increase both the winning probability and the payoff in the second stage. Our conjecture is that coalitional payoff has a non-monotonic evolution because there are two effects that sometime work together and sometimes not.

This result has therefore a corollary:

Corollary 13 *For the exclusion game $v_\alpha(S) = v_\beta(S)$ if $n \leq (m+1)(1-p^S)\frac{l}{r} + 2$ for any $S \subset N$.*

Proof of the Corollary. Simple inspection of (2) show us that the worst case scenario for members of S when they are waiting for the choice of their rivals occurs when r^{-S} attains its maximum. When $m \geq 1$ the coalition $N \backslash S$ must form and all its members must put their entire endowment; then $\widehat{r}^{-S} = (n-s)^m$.When $m < 1$, the complement of S must form singletons, put also $r_i = 1$ and then $\widehat{r}^{-S} = (n-s)$.

Let us suppose that we are in the former case (the other is analogous). Then, the alpha characteristic function is just the best response to $(n-s)^m$.

We know by Lemma 12 that the indirect characteristic function is decreasing in r^{-S} if the coalitional size is the threshold above. Given that $v_\alpha(N) = v_\beta(N)$ always coincide we need just $n - 1 \leq (m+1)(1-p^S)\frac{l}{r} + 1$ to ensure that for any coalition $u_i^*(r^{-S})$ attains its minimum also when $r^{-S} = (n-s)^m$. So finally we have:

$$v_\alpha(S) = u_i^S(r^S((n-s)^m), (n-s)^m) = u_i^*((n-s)^m) = \underset{r^{-S}}{Min}\ u_i^*(r^{-S}) = v_\beta(S)$$

■

Hence we must restrict the coincidence result, which is complete for Common-Pool games (Meinhardt, 1999) and Cournot games (Zhao, 1999), in our case: When there are not too many players, the effectivity of conflict or the ratio $\frac{l}{r}$ (not too concave technologies) are high enough, players are indifferent between reacting passively to the outsiders' choice that best punishes the coalition and waiting for the punishment when they employ best responses.

The next question is if, under the current assumptions about outsiders' behavior, there is some room for cooperation, i.e. if the payoff received when the grand coalition forms cannot be blocked by the one received inside a subcoalition.

Definition 14 *The α-core (β-core) exclusion game is nonempty if there exist no empty coalition $S \subset N$ such that $v_\alpha(S) > v_\alpha(N)$ $(v_\beta(S) > v_\beta(N))$.*

It is precisely the extreme kind of responses assumed by the alpha and beta characteristic functions what triggers the non-emptiness of the α-core (and coincidence therefore in the cases mentioned in Lemma 12 with the β-core).

Scarf (1971) stated that the NTU α-core is non-empty if the strategy space for each player is compact and convex and payoff functions are all continuous and quasiconcave. This conditions are satisfied by our game (recall that payoff function is strictly concave).

Corollary 15 *The exclusion game has nonempty $\alpha-$core. Moreover, it has nonempty β core as well if $n \leq (m+1)(1-p^S)\frac{l}{r} + 2$ for any $S \subset N$.*

Two Remarks:

(a) The grand coalition is stable in the shadow of exclusion if players are committed to inflict as much harm as possible to potential deviators (implied by the alpha stability notion). Whereas the formation of the grand coalition implies a peaceful access, the severe punishment suffered by coalitions outweights the potential gains derived from a smaller number of agents exploiting the resource.

(b) Corollary 15 does not imply that the β core is empty. In fact, if $\widehat{r}^{-S} = (n-s)^m$ is the actual minimizer of u_i^*, the Exclusion game satisfies all the conditions posed in Theorem 1 in Zhao (1999) for the non-emptiness of the β core. The real problem is that it is not possible in this framework to compare a corner solution with possible interior minimizers.

However, this cooperation is based on threats, not in a surplus produced by acceding together. As we will se now, it would be better from the social point of view that a coalition acceded to the common.

3.2 Sequential approach

The static approach to coalition formation has been criticized because it assumes an outsiders' behavior that may be completely irrational: It cannot be worthy for agents in $N \backslash S$ to punish S in a "bloodthirsty" manner, that is, as much as they can. In our game, this point becomes even more evident given that it is never optimal for players to invest their entire endowment in exclusionary purposes. Hence, we should allow the complement of S to employ just best responses: Coalitions will be able therefore to attach a payoff to each structure that may arise and predict in some extent the reply of outsiders to its movements.

However, there is no a unique approach to this issue. Several rules of coalition formation has been proposed for the analysis of endogenous coalition structures under externalities. Here, we will follow Bloch (1996) approach, where coalitions form if and only if all members agree to do it *à la* Runbinstein: The first player in the rule of order makes a proposal for a coalition; the players that form this proposed coalition decide sequentially to accept or not that proposal. The process stops when all members accept or one rejects. In the former case, the coalition finally forms; in the latter, the rejector must make another proposal. The author shows that this game yields the same stationary subgame perfect equilibrium coalition structure as the much simpler "Size Announcement game": First player proposes a coalition of size s_1 that immediately forms. Then the $(s_1 + 1)$-th in the rule of order proposes a coalition s_2 and so on, until the player set is exhausted. The game is solved through backward induction and has generally a unique subgame perfect equilibrium.

Hence, in order to know if players will decide or not to form a coalition, we need to know what happens with individual payoffs as size increases.

Proposition 16 *Given a level of outsiders' effort r^{-S}, the indirect payoff function is decreasing in size if $s > m(1 - p^S)\frac{l}{r} + 1$.*

The intuition of this result is simple: There is a critical mass such that the coalition can gain the access to the resource with so high probability that this effect can not counterpart the strong overexploitation implied by a final level of total labor that implies very close to $l^*(s)$

The overxploitation treshold yields a higher bound for coalitional sizes in a given coalition structures.

Proposition 17 *All coalitions with size bigger than $\frac{m}{Min_s r(s)^{\min}} + 1$ cannot belong to a stable coalition structure.*

Proof. The treshold (**??**) is smaller than $\frac{m}{r} + 1$. Hence, we can be sure that all coalitions bigger than $\frac{m}{Min_s r(s)^{\min}} + 1$ will underexploit the resource, and therefore, the indirect payoff function will be increasing in size. And this for any possible level of r^{-S} and any coalitional size. ■

Notice that the previous proposition ensures that the formation of the grand coalitions is never a stable coalition structure if $n > \frac{m}{Min_s r(s)^{\min}} + 1$.

Moreover, if $m \leq Min_s\ r(s)^{\min}$ we can extend the result to any coalition bigger than singleton. In such case, players find worthless to merge, both because the decreasing returns to scale of conflict effort and because marginal productivity of conflict activities is so low that overxploitation is a very likely outcome. In particular for the production function $f(l^S) = al^S - b(l^s)$, this condition reduces to $m \leq 1 - \frac{\theta}{2}$, where $\theta = \frac{a-c}{b}$, a approximate measure of the linearity of the technology.

What happens with high levels of m? Out conjecture is that there are always a set of (high) values of m and a of (high) elasticities that support the formation of just two coalitions. Moreover, we think that as m increases, stable coalition structures tend to consist of an asymmetric couple of coalitions. It is not clear that this factor will work in favor of the formation of the grand coalition, given that, although expenditures on conflict will be substantially higher, size gains are huge as long as external threat exists. Another condition would be a sufficiently linear technology in order to make the overxploitation implied by the formation of $\{N\}$ a good alternative to strong hostilities.

4 A second example

The initial data of this game is taken from Meinhardt (1999). This will allow us to compare the exclusion game with Common pool games: Assume that $N = 4$, the players have 35 units to spend in production or exclusion and the unit cost of labor is 3. The production function takes the form $f(l) = 23l - \frac{1}{8}l^2$. The final ingredient is the conflict technology that is assumed to be of the form (1) where we will allow m to be 1 or 2. Then, the alpha and beta characteristic functions when $m = 1$:

$$v_\alpha(\{i\}) = 44,\ v_\alpha(\{ij\}) = 160,\ v_\alpha(\{ijk\}) = 327.9,\ v_\alpha(N) = 512$$

and when $m = 2$:

$$v_\alpha(\{i\}) = 10.57,\ v_\alpha(\{ij\}) = 106,\ v_\alpha(\{ijk\}) = 379.5,\ v_\alpha(N) = 512$$

These values are below Meinhardt's one. This suggest that economic punishments (overharvesting of the fishery for instance) are less severe that non-economic ones (possibility of exclusion). However, contrary to his game, ours, so defined, fails to be convex.

When we assume that coalitions play best responses we should employ the partition function, so we express individual expected payoffs by $u_i(s, \pi\backslash\{S\})$. When $m = 1$

$$\begin{aligned} u_i(1, \{\{j\}\{k\}\{l\}\}) &= 82.84,\ u_i(1, \{\{j\}\{kl\}\}) = 102,\ u_i(1, \{jkl\}) = 155.63 \\ u_i(2, \{\{k\}\{l\}\}) &= 137.8,\ u_i(2, \{kl\}) = 176.8 \quad u_i(3, \{l\}) = 151.63 \end{aligned}$$

and when $m = 2$:

$$\begin{aligned} u_i(1,\{\{j\}\{k\}\{l\}\}) &= 60.63,\ u_i(1,\{\{j\}\{kl\}\}) = 66.75,\ u_i(1,\{jkl\}) = 94.35 \\ u_i(2,\{\{k\}\{l\}\}) &= 142.13,\ u_i(2,\{kl\}) = 158.14 \quad u_i(3,\{l\}) = 170.54 \end{aligned}$$

What would be the outcome of a sequential coalition formation process? Bloch's game yields a unique structure, the symmetric $\{\{ij\}\{kl\}\}$ when $m = 1$ and $\{\{ijk\}\{l\}\}$ when $m = 2$. The reason for this lies at the fact that when $m = 2$ the three players coalition is overexploiting. Then, the payoff would increase if they expel one player because they can employ conflict as a discipline device. However, small coalitions want to absorb as many contraries as possible because winning probability and production payoff increase.

Their impact on efficiency is different: Notice that when $m = 1$ individual payoffs are aligned with efficiency because $\{\{ij\}\{kl\}\}$ is the more efficient structure: the sum of coalitional payoffs is the largest. However, when $m = 2$ players in $\{ijk\}$ can weaken so much the remaining player that the individual payoff is the largest possible. However, the symmetric two-coalition structure is the efficient one.

Efficiency calls for deterring players from the access to the resource; conflict is socially good at this point, when forces are even. But, when returns of conflict are big, large coalitions become profitable because it is relatively cheap to keep outsiders away.

Notice that conflict is anyway two-sided. Such result would shed light on the possible foundations of conflict models, typically modelled as two-player contest, although as we have seen, they may not need to be symmetric

Finally, notice that the possibility of conflict alleviates partially the 'tragedy of the commons': (i) The number of players that finally accede is lower (ii) the expected production is closer to the joint exploitation of the resource, the best case scenario (there the production is 800; with exclusion it may be of 632).

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A Appendix

Proof of Lemma 4. Let $h(s,c,\varepsilon) = f'(l^*(s)) + (s-1)f(l^*(s))/l^*(s) - sc = 0$. By the implicit function theorem:

$$\frac{\partial l^*(s)}{\partial s} = \frac{-\partial h(s,c,\varepsilon)/\partial s}{\partial h(s,c,\varepsilon)/\partial l^*(s)}$$

By simple calculations

$$\frac{\partial h(s,c,\alpha)}{\partial l^*(s)} = f''(l^*(s)) + \frac{(s-1)}{l^*(s)}(f'(l^*(s)) - \frac{f(l^*(s))}{l^*(s)}) < 0 \qquad \text{by concavity}$$

It is clear that the derivative of $h(s,c,\alpha)$ with respect to s and c are positive and negative respectively. Hence $\frac{\partial l^*(s)}{\partial s} > 0$

$$\frac{\partial r^{Min}}{\partial s} = -\frac{\partial (l^*(s)/s)}{\partial s} = -[\frac{\partial l^*(s)}{\partial s}\frac{1}{s} - \frac{l^*(s)}{s^2}] \qquad \text{some manipulation yields}$$

$$= -\frac{l^*(s)}{s^2}[s\frac{f(l^*(s))/l^*(s) - c}{(s-1)[f(l^*(s))/l^*(s) - f'(l^*(s))] - f''(l^*(s))l^*(s)} - 1]$$

by condition (3) we know that

$s[f(l^*(s))/l^*(s) - c] = f(l^*(s))/l^*(s) - f'(l^*(s))$

and then

$\frac{\partial r^{\min}}{\partial s} = -\frac{l^*(s)}{s^2}[\frac{f(l^*(s))/l^*(s) - f'(l^*(s))}{(s-1)[f(l^*(s))/l^*(s) - f'(l^*(s))] - f''(l^*(s))l^*(s)} - 1] > 0$

For any $s \geq 2$ the numerator is greater than the denominator and the first term in brackets is smaller than one. Then the overall expression is positive.

Finally if we rearrange condition (3), one obtains:

$$h(s, c, \varepsilon) = (s + \varepsilon - 1)\frac{f(l^S)}{l^S} - sc = 0$$

and

$\frac{\partial r^{Min}}{\partial \varepsilon} = \frac{1}{s}\frac{\partial h(s,c,\varepsilon)/\partial \varepsilon}{\partial h(s,c,\varepsilon)/\partial l^*(s)} < 0$

However, we can only establish a partial ordering because we cannot compare situations in which elasticities' rankings change as total labor input varies. ■

Proof of Lemma 5. The Kuhn-Tucker conditions of the maximization problem faced by an individual are:

(i) $\frac{\partial \mathcal{L}}{\partial r_i} = \frac{\partial u_i^S}{\partial r_i} + \mu_1 - \mu_2 = 0$

(ii) $r_i \geq 0,\ 1 - r_i \geq 0$

(iii) $\mu_1 \geq 0, \mu_2 \geq 0$

(iv) $\mu_1 r_i = 0$

(v) $\mu_2(1 - r_i) = 0$

The second part of the statement can be obtained by simple differentiation of the objective function provided that $\mu_1 = \mu_2 = 0$.

Now, by Lemma 1, we know that the second constraint (i.e. $r_i \leq 1$) is never binding. Then, condition (i) reduces to $\frac{\partial u_i^S}{\partial r_i} \leq 0$. Knowing this fact, we can establish uniqueness of the best reply by means of the following auxiliar Lemma:

Lemma 18 (A1) *If* $\frac{\partial u_i^S}{\partial r_i} \leq 0$ *then the objective function is strictly concave (i.e.,* $\frac{\partial u_i^S}{\partial^2 r_i} < 0$*)*

Proof. If $\frac{\partial u_i^S}{\partial r_i} \leq 0$ then

$$\frac{m(sr)^{-1}r^{-S}}{(sr)^m + r^{-S}}[\frac{1}{s}f(s-sr) - c(1-sr)] \leq \frac{1}{s}[f'(s-sr) + (s-1)\frac{f(s-sr)}{s-sr} - sc].$$

Multiply now both sides by $-2\frac{m(sr)^{m-1}r^{-S}}{[(sr)^m+r^{-S}]^2}$ then

$$\begin{aligned} & -2\frac{m^2(sr)^{m-2}(r^{-S})^2}{[(sr)^m + r^{-S}]^3}[\frac{1}{s}f(s-sr) - c(1-r)] \\ > & -2\frac{m(sr)^{m-1}r^{-S}}{[(sr)^m + r^{-S}]^2}\frac{1}{s}[f'(s-sr) + (s-1)\frac{f(s-sr)}{s-sr} - sc]. \quad (12) \end{aligned}$$

The second derivative (after applying symmetry) is:

$$\begin{aligned} \frac{\partial u_i^S}{\partial^2 r_i} = & [\frac{m^2(sr)^{m-2}(r^{-S})^2}{[(sr)^m + r^{-S}]^3} - \frac{m(m+1)(sr)^{m-2}(r^{-S})}{[(sr)^m + r^{-S}]^2}][\frac{1}{s}f(s-sr^*) - c(1-r^*)] - \\ & -2\frac{m(sr)^{m-1}r^{-S}}{[(sr)^m + r^{-S}]^2}\frac{1}{s}[f'(s-sr) + (s-1)\frac{f(s-sr)}{s-sr} - sc] - \\ & -\frac{(sr)^m}{(sr)^m + r^{-S}}\frac{1}{s}[2(s-1)\frac{1}{s-sr}(\frac{f(s-sr)}{s-sr} - f'(s-sr)) - f''(s-sr)]. \end{aligned}$$

If we substitute we get

$$\begin{aligned} \frac{\partial u_i^S}{\partial^2 r_i} < & [\frac{1}{s}f(s-sr) - c(1-r)](-\frac{m(m+1)(sr)^{m-2}(r^{-S})}{[(sr)^m + r^{-S}]^2} - \frac{m^2(sr)^{m-2}(r^{-S})^2}{[(sr)^m + r^{-S}]^3}) \quad (13) \\ & -\frac{(sr)^m}{(sr)^m + r^{-S}}\frac{1}{s}[2(s-1)\frac{1}{s-sr^*}(\frac{f(s-sr)}{s-sr} - f'(s-sr)) - f''(s-sr)]. \end{aligned}$$

The first term is negative , and given that average productivity is below marginal, the second term is negative too. ■

And given that the objective function is strictly concave, there is a unique maximizer. ■

Proof of Lemma 6. Again, by the implicit function theorem

$\frac{\partial r_i(r^{-S})}{\partial r^{-S}} = \frac{-\partial h(r^S, r^{-S})/\partial r^{-S}}{\partial h(r^S, r^{-S})/\partial r^S}$

by differentiating condition (7) one gets

$$\begin{aligned} -\frac{\partial h(r^S, r^{-S})}{\partial r^{-S}} = & -\frac{m(r^S)^{m-1}}{[(r^S)^m + r^{-S}]^3}((r^S)^m - r^{-S})[f(s-r^S) - c(s-r^S)] - \\ & -\frac{(r^S)^m}{[(r^S)^m + r^{-S}]^2}[(s-1)\frac{f(s-r^S)}{s-r^S} + f'(s-r^S) - sc] \end{aligned}$$

And, by first order conditions

$$-\frac{\partial h(r^S, r^{-S})}{\partial r^{-S}} = -\frac{m(r^S)^{m-1}}{[(r^S)^m + r^{-S}]^3}(r^S)^m[f(s-r^S) - c(s-r^S)] < 0.$$

And given that the objective function is concave $\frac{\partial h(r^S, r^{-S})}{\partial r^S} < 0$ and therefore $\frac{\partial r_i(r^{-S})}{\partial r^{-S}} > 0$.

For the second part of the statement we need the complete expression for the derivative. In order to save notation, let us denote f by $f(s - r^S)$ and so on:

$$\frac{\partial r_i(r^{-S})}{\partial r^{-S}} = \frac{1}{s} \frac{-\frac{m(r^S)^{m-1}}{[(r^S)^m + r^{-S}]^2}[f - c(s - r^S)]}{\frac{\partial h(r^S, r^{-S})}{\partial r}}$$

where

$$\frac{\partial h(r^S,r^{-S})}{\partial r} = \frac{mr^{-S}(r^S)^{-2}[(m-1)r^{-S} - (m+1)(r^S)^m]}{[(r^S)^m + r^{-S}]^2}[f - c(s - r^S)] - \frac{mr^{-S}(r^S)^{-1}}{(r^S)^m + r^{-S}}[f' - c] -$$

$$-\frac{mr^{-S}(r^S)^{-1}}{(r^S)^m + r^{-S}}[(s-1)\frac{f}{s - r^S} + f' - sc] + [f'' + \frac{(s-1)}{1 - r_i}(f' - \frac{f}{(s - r^S)})]$$

By first order condition (7) we know both that

(i) $\frac{mr^{-S}(r^S)^{-1}}{(r^S)^m + r^{-S}}[f' + (s-1)\frac{f}{s - r^S} - sc] = \frac{m^2 (r^S)^{-2}(r^{-S})^2}{[(r^S)^m + r^{-S}]^2}[f(s - r^S) - c(s - r^S)]$,

(ii) $-\frac{mr^{-S}(r^S)^{-1}}{(r^S)^m + r^{-S}}[f' - c] = \frac{mr^{-S}(r^S)^{-1}}{(r^S)^m + r^{-S}}[\frac{s-1}{s - r^S} - \frac{mr^{-S}(r^S)^{-1}}{(r^S)^m + r^{-S}}][f - c(s - r^S)]$ and that

(iii) $\frac{(s-1)}{s - r^S}(f' - \frac{f}{(s - r^S)}) = \frac{(s-1)}{s - r^S}[f(s - r^S) - c(s - r^S)][\frac{mr^{-S}(r^S)^{-1}}{(r^S)^m + r^{-S}}\frac{1}{s} - \frac{1}{s - r^S}]$.

So, the expression reduces to

$$\frac{\partial r_i(r^{-S})}{\partial r^{-S}} = \frac{1}{s}\frac{1}{(sr)^m + r^{-S}} \frac{\frac{m(sr)^{m-1}}{(r^S)^m + r^{-S}}[f(s - sr) - c(s - sr)]}{\frac{m(sr)^{-1}r^{-S}}{(sr)^m + r^{-S}}[\frac{m+1}{sr} + \frac{(s-1)}{1-r}\frac{r}{1-r}\frac{1}{1-ps}\frac{1}{m} - 2\frac{s-1}{s - sr}][f - c(s - sr)] - f''} \tag{14}$$

and simple inspection of the expression of the derivative shows that its slope is decreasing in r^{-S} ∎

Proof of Lemma 10. By the Implicit Function Theorem

$\frac{\partial r_i}{\partial s} = \frac{1}{s}\frac{-\partial h(s)/\partial s}{\partial h(s)/\partial r_i}$ where $h(s)$ is given by (7).

$$-\frac{\partial h(s)}{\partial s} = -r\varphi(s) + \frac{(sr)^m}{(sr)^m + r^{-S}}[\frac{f}{s - sr} - c] - \frac{mr^{-S}(sr)^{m-1}}{[(sr)^m + r^{-S}]^2}[f' - c] +$$

$$+\frac{(sr)^m}{(sr)^m + r^{-S}}[f'' + \frac{(s-1)}{1 - r_i}(f' - \frac{f}{(s - r^S)})]$$

where

$$\varphi(s) = \frac{mr^{-S}(sr)^{m-2}[(m-1)r^{-S} - (m+1)(sr)^m]}{[(sr)^m + r^{-S}]^3}[f - c(s - r^S)] - \frac{mr^{-S}(sr)^{m-1}}{[(sr)^m + r^{-S}]^2}[f' - c] -$$

$-\frac{mr^{-S}(sr)^{m-1}}{[(sr)^m+r^{-S}]^2}[(s-1)\frac{f}{s-sr}+f'-sc]+\frac{(sr)^m}{(sr)^m+r^{-S}}[f''+\frac{(s-1)}{1-r_i}(f'-\frac{f}{(s-r^S)})]=\frac{\partial h(s)}{\partial r}\frac{1}{s}$

Then, dividing by $\frac{\partial h(s)}{\partial r_i}$ we get

$$\frac{\partial r_i}{\partial s}=-\frac{r}{s}+\frac{1}{s}\frac{\frac{(sr)^m}{(sr)^m+r^{-S}}[\frac{f}{s-sr}-c]-\frac{mr^{-S}(sr)^{m-1}}{[(sr)^m+r^{-S}]^2}[f'-c]+\frac{(sr)^m}{(sr)^m+r^{-S}}[f''+\frac{(s-1)}{1-r_i}(f'-\frac{f}{(s-r^S)})]}{\varphi(s)} \tag{15}$$

By first order conditions we can reduce both the numerator and the denominator of the second term of (15) and one finally obtains

$$\frac{\partial r_i}{\partial s}=-\frac{r}{s}+\frac{1}{s}\frac{\frac{m}{sr}(1-p^S)[2\frac{s-1}{s-sr}-\frac{m}{sr}(1-p^S)-[\frac{(s-1)}{1-r}-1]\frac{r}{1-r}\frac{1}{m}\frac{1}{1-ps}][f-c(s-r^S)]+f''}{\frac{m}{sr}(1-p^S)[2\frac{s-1}{s-sr}-\frac{m+1}{sr}-\frac{(s-1)}{1-r}\frac{r}{1-r}\frac{1}{1-ps}\frac{1}{m}][f-c(s-sr)]+f''}$$

It can be seen that in absolute terms the slope is bigger the smaller the coalition

Then $\frac{\partial r_i}{\partial s}>0$ if and only if

$$\frac{m}{sr}(1-p^S)[2\frac{s-1}{s}-\frac{m}{sr}(1-p^S)-(s-2)\frac{r}{1-r}\frac{1}{m}\frac{1}{1-ps}+\frac{m+1}{s}][f-c(s-r^S)]+(1-r)f''<0$$

A sufficient condition when $s\geq 2$ is that

$$2-\frac{m}{s}(1-p^S)-(s-2)\frac{r}{1-r}\frac{1}{m}+\frac{m-1}{s}<0$$

It is evident that this expression is decreasing in s. The question is if it there exists a size that makes it negative. It turns out that for values above

$$s\geq 2+m(1-p^S)\frac{(1-r)}{r}(1+\sqrt{1+\frac{r}{1-r}\frac{m+1}{m}})$$

the whole derivative is negative. Then s^* is below this value. ■

Proof of Lemma 12. Let r' be the short hand notation of $\frac{\partial r_i}{\partial r^{-S}}$ and $r(r^{-S})$ the best response strategy of the member of S.Then indirect payoff function and its derivative with respect to r^{-S} are

$$u_i^*(r^{-S})=u_i^S(sr(r^{-S}),r^{-S})=\frac{(sr(r^{-S}))^m}{(sr(r^{-S}))^m+r^{-S}}[\frac{1}{s}f(s-s(r^{-S}))-c(1-r(r^{-S}))]$$

$$\frac{\partial u_i^*(r^{-S})}{\partial r^{-S}} = \frac{\partial u_i^*}{\partial r}r' + \frac{\partial u_i^*}{\partial r^{-S}} = p^S[r'\frac{s-1}{1-r} - \frac{1}{(sr(r^{-S}))^m + r^{-S}}][f - c(s - sr(r^{-S}))]$$

that has no clear sign given that $r' > 0$. Given what we know by Lemma 6,

$$\frac{\partial u_i^*(r^{-S})}{\partial r^{-S}} = p^S\frac{[f-c(s-sr)]}{(sr)^m + r^{-S}}[\frac{\frac{m(sr)^{m-1}}{(sr)^m+r^{-S}}[f-c(s-sr)]}{\frac{m(sr)^{-1}r^{-S}}{(sr)^m+r^{-S}}[\frac{m+1}{s-1}\frac{1-r}{r} + \frac{sr}{1-r}\frac{1}{1-ps}\frac{1}{m} - 2][f-c(s-sr)] - \frac{s-sr}{s-1}f''} - 1]$$

Unfortunately we should restrict to conditions that ensure the negative sign of this derivative. It is sufficient to show that

$\frac{1-r}{r}\frac{m+1}{s-1} + \frac{sr}{1-r}\frac{1}{1-p^S}\frac{1}{m} - 2 - \frac{(sr)^m}{r^{-S}} > 0$

Once here we can take advantage from the point (*iii*) in Lemma 6, and given that marginal productivity is below average one $\frac{m}{sr}(1-r)(1-p^S) < 1$, so the expression above is greater than$\frac{1-r}{r}\frac{m+1}{s-1} - \frac{1}{1-p^S}$ and a coalitional size that makes it grater than zero is $s < (m+1)(1-p^S)\frac{l}{r} + 1$. ■

Proof of Proposition 16. Let r' be the short hand notation of $\frac{\partial r_i}{\partial s}$ and $r(s)$ the best response strategy of the member of S.Then indirect payoff function and its derivative after some algebra with respect to s are

$$u_i^*(s) = u_i^S(sr(s), r^{-S}) = \frac{(sr(s))^m}{(sr(s))^m + r^{-S}}[\frac{1}{s}f(s - sr(s)) - c(1 - r(s))]$$

$$\frac{\partial u_i^*(s)}{\partial s} = \frac{\partial u_i^*}{\partial r}r' + \frac{\partial u_i^*}{\partial s} = \frac{p^S}{s}[\frac{m}{sr}(1-p^S) - \frac{s-r}{1-r} + r'\frac{s-1}{1-r}][f - c(s - sr(r^{-S}))]$$

Suppose now that the coalition overexploits, that is $\frac{m}{sr}(1-p^S) < \frac{s-1}{1-r}$. We know also that $r' < \frac{1-r}{s}$ so then

$\frac{\partial u_i^*(s)}{\partial s} < \frac{p^S}{s}[\frac{s-1}{s-sr} - \frac{s-r}{1-r} + \frac{s-1}{s}][f - c(s - sr(r^{-S}))] = \frac{p^S}{s}[\frac{2s-2-s^2+r}{s(1-r)}][f - c(s - sr(r^{-S}))]$

that is negative for any s

Now, with underexploitation

$\frac{\partial u_i^*(s)}{\partial s} > \frac{p^S}{s}[\frac{s-1}{s-sr} - \frac{s-r}{1-r} + r'\frac{s-1}{1-r}][f - c(s - sr(r^{-S}))]$ and given that $r' < \frac{1}{s}$ one obtains

$\frac{\partial u_i^*(s)}{\partial s} > \frac{p^S}{s(s-sr)}[(s-1)(1-s(1-r')) - s(1-r)][f - c(s - sr(r^{-S}))]$

: ■

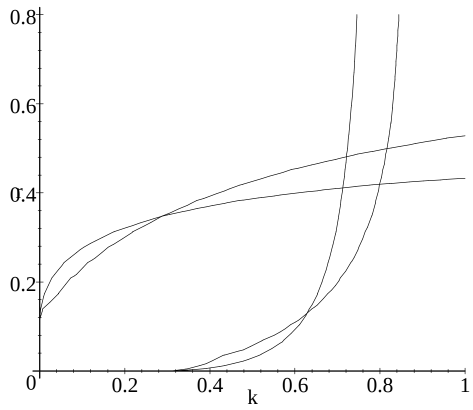

Fig. 1a: The effect of technology.

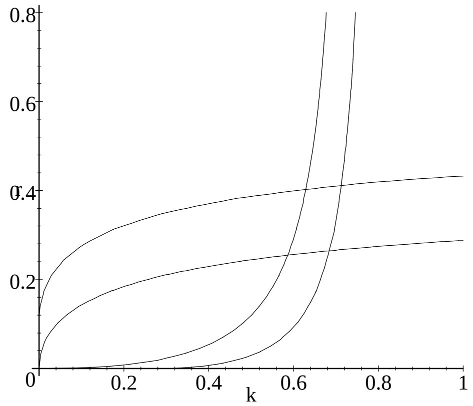

Fig. 1b: The effect of conflict effectiveness.

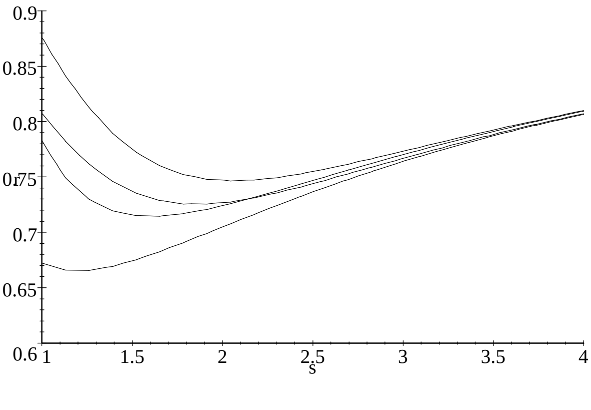

Fig 2: The effect of size